Robotics - Projective Geometry and Camera model

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Outline

Outline

Projective Transformations - Recall

Projective Transformation

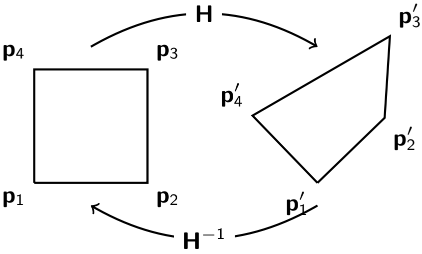
$$\mathbf{x}' = \mathbf{H}\,\mathbf{x}$$

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

Notes

- Map plane to plane
- It's a linear transformation

  in homogeneous coordinates
- It's homogeneous too $\lambda\mathbf{H} \equiv \mathbf{H}$



Projective Transformations - Image Rectification - 1

Homography Estimation

- Take four point on first image $\mathbf{x}_i = \left[x_i,\ y_i,\ w_i\right]^T$
- Map on four known destination points $\mathbf{x}'_i = \left[x'_i,\ y'_i\right]^T$
- Rewrite: $\begin{cases} x''_i = h_{11}x_i + h_{12}y_i + h_{13}w_i \\ y''_i = h_{21}x_i + h_{22}y_i + h_{23}w_i \\ w''_i = h_{31}x_i + h_{32}y_i + h_{33}w_i \end{cases}$
- In cartesian: $\begin{cases} x'_i = \frac{h_{11}x_i + h_{12}y_i + h_{13}w_i}{h_{31}x_i + h_{32}y_i + h_{33}w_i} \\ y'_i = \frac{h_{21}x_i + h_{22}y_i + h_{23}w_i}{h_{31}x_i + h_{32}y_i + h_{33}w_i} \end{cases}$
- Fix $h_{33} = 1$ and rewrite $\begin{cases} x'_i(h_{31}x_i + h_{32}y_i + w_i) = h_{11}x_i + h_{12}y_i + h_{13}w_i \\ y'_i(h_{31}x_i + h_{32}y_i + w_i) = h_{21}x_i + h_{22}y_i + h_{23}w_i \end{cases}$

Projective Transformations - Image Rectification - 2

- Expand and separate $\begin{cases} x_i h_{11} + y_i h_{12} + w_i h_{13} - x_i' x_i h_{31} - x_i' y_i h_{32} & = & x_i' w_i \\ x_i h_{21} + y_i h_{22} + w_i h_{23} - y_i' x_i h_{31} - y_i' y_i h_{32} & = & y_i' w_i \end{cases}$
- Matrix form (2-lines for each point)

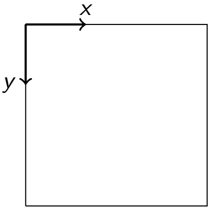
$$\begin{bmatrix} x_1 & y_1 & w_1 & 0 & 0 & 0 & -x_1' x_1 & -x_1' y_1 \\ 0 & 0 & 0 & x_1 & y_1 & w_1 & -x_1' x_1 & -x_1' y_1 \\ x_2 & y_2 & w_2 & 0 & 0 & 0 & -x_2' x_2 & -x_2' y_2 \\ 0 & 0 & 0 & x_2 & y_2 & w_2 & -x_2' x_2 & -x_2' y_2 \\ x_3 & y_3 & w_3 & 0 & 0 & 0 & -x_3' x_3 & -x_3' y_3 \\ 0 & 0 & 0 & x_3 & y_3 & w_3 & -x_3' x_3 & -x_3' y_3 \\ x_4 & y_4 & w_4 & 0 & 0 & 0 & -x_4' x_4 & -x_4' y_4 \\ 0 & 0 & 0 & x_4 & y_4 & w_4 & -x_4' x_4 & -x_4' y_4 \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \end{bmatrix} = \begin{bmatrix} x_1' w_1 \\ y_1' w_1 \\ x_2' w_2 \\ y_2' w_2 \\ x_3' w_3 \\ y_3' w_3 \\ x_4' w_4 \\ y_4' w_4 \end{bmatrix}$$

- System $\mathbf{Ax} = \mathbf{b}$ e.g. in Matlab solved with $\mathbf{x} = \mathbf{A} \backslash \mathbf{b}$

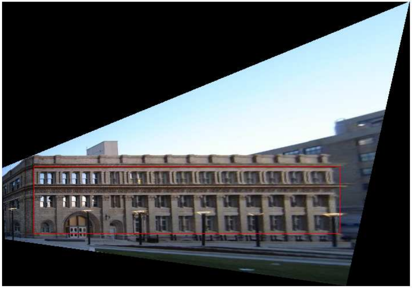
Projective Transformations - Image Rectification - Example

Original Image

Image reference system



- $\mathbf{x}$: {179,525}, {187,73}, {690,307}, {698,467}
- $\mathbf{x}'$: {0,180}, {0,0}, {822,0}, {822,180}
- $$\mathbf{H} = \begin{bmatrix} 0.4659 & 0.0082 & -87.7293 \\ -0.1573 & 0.3382 & 4.7322 \\ -0.0011 & 0.0001 & 1.0000 \end{bmatrix}$$



Projective Transformations - Lines and Conics

Points

$$\mathbf{p}' = \mathbf{H}\mathbf{p}$$

Lines

$$\mathbf{l}' = \mathbf{H}^{-T}\mathbf{l}$$

Proof

- $\mathbf{l}^T\mathbf{p} = 0$
- $\mathbf{l}'^T\mathbf{p}' = 0$
- $\mathbf{l}'^T\mathbf{H}\mathbf{p} = 0$
- $(\mathbf{H}^{-T}\mathbf{l})^T\mathbf{H}\mathbf{p} = 0$
- $\mathbf{l}^T\mathbf{H}^{-1}\mathbf{H}\mathbf{p} = 0$

Conics

$$\mathbf{C}' = \mathbf{H}^{-T}\mathbf{C}\mathbf{H}^{-1}$$

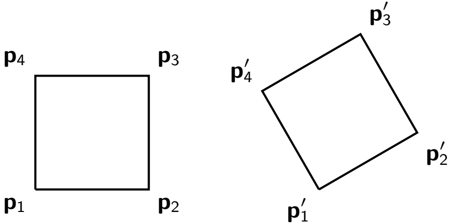
Proof

- $\mathbf{p}^T\mathbf{C}\mathbf{p} = 0$
- $\mathbf{p}'^T\mathbf{C}'\mathbf{p}' = 0$
- $(\mathbf{H}\mathbf{p})^T\mathbf{H}^{-T}\mathbf{C}\mathbf{H}^{-1}\mathbf{H}\mathbf{p} = 0$

Outline

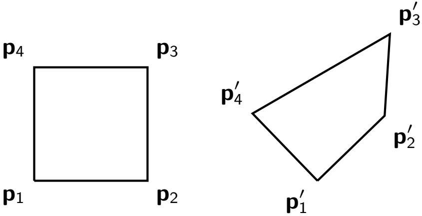
Transformations - Recall

Rototranslation



$$\mathbf{p}' = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \mathbf{p}'$$

Homography

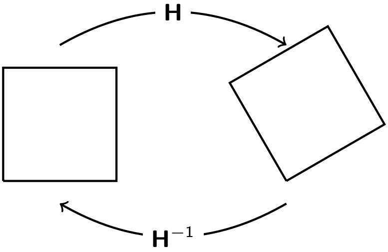


$$\mathbf{p}' = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \mathbf{p}'$$

Class I - Isometries - i.e., Rototranslations

$$\mathbf{p}' = \begin{bmatrix} \xi \cos(\theta) & -\sin(\theta) & \mathbf{t}_x \\ \xi \sin(\theta) & \cos(\theta) & \mathbf{t}_y \\ 0 & 0 & 1 \end{bmatrix} \mathbf{p}'$$

- *iso*: same, *metric*: measure
- $\xi = +1$ orientation preserving
- $\xi = -1$ orientation reversing
- 3 DoF (2 translation, 1 rotation)
- Special cases:
  - Pure rotation
  - Pure translation



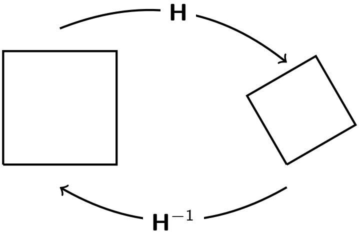
Invariants

- Length
- Area
- Angle

Class II - Similarities

$$\mathbf{p}' = \begin{bmatrix} s\cos(\theta) & -s\sin(\theta) & \mathbf{t}_x \\ s\sin(\theta) & s\cos(\theta) & \mathbf{t}_y \\ 0 & 0 & 1 \end{bmatrix} \mathbf{p}'$$

- Isometry + scale factor
- 4 DoF (2 translation, 1 rotation, 1 scale)
- $det(s\mathbf{R}) = s$



<u>Invariants</u>

- Shape
- Ratios of length
- Ratios of areas
- Angle
- Parallel lines

Class III - Affine transformations

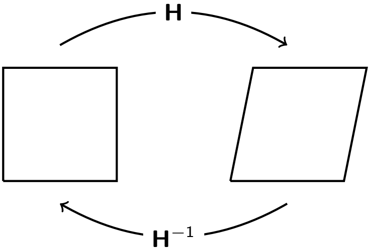
$$\mathbf{p}' = \begin{bmatrix} a_{11} & a_{11} & \mathbf{t}_x \\ a_{21} & a_{22} & \mathbf{t}_y \\ 0 & 0 & 1 \end{bmatrix} \mathbf{p}'$$

- Non-isotropic scaling
- 6 DoF

  (2 translation, 2 rotation, 2 scale)
- $\mathbf{A} = \begin{bmatrix} a_{11} & a_{11} \\ a_{21} & a_{22} \end{bmatrix} = \mathbf{UDV}^T$
- $\mathbf{UDV}^T = \left(\mathbf{UV}^T\right)\left(\mathbf{VDV}^T\right)$

  $\mathbf{U}$, $\mathbf{V}$ orthogonal, $\mathbf{D}$ diagonal
- $\mathbf{R}(\theta)\left(\mathbf{R}(-\phi)\mathbf{D}\mathbf{R}(\phi)\right)$

  rotation on scaled axis



Invariants

- Parallel lines
- Ratios of parallel segment lengths
- Ratios of areas

Class IV - Homographies

$$\mathbf{p}' = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \mathbf{p}'$$

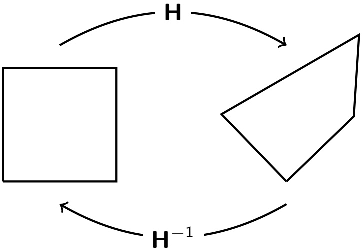
- Mapping plane to plane

  linear in homogeneous coordinates

- 8 DoF

  2 translation, 2 rotation,

  2 scale, 2 for $\mathbf{l}\infty$



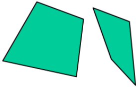
Invariants

- Collinearities
- Cross-ratio of four points on a line

2D Transformations overview

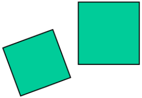
Projective
8dof

$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$



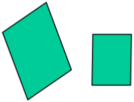
Similarity
4dof

$$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



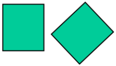
Affine
6dof

$$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



Euclidean
3dof

$$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



	Euclidean	similarity	affine	projective
Transformations				
rotation	X	X	X	X
translation	X	X	X	X
uniform scaling		X	X	X
nonuniform scaling			X	X
shear			X	X
perspective projection				X
composition of projections				X

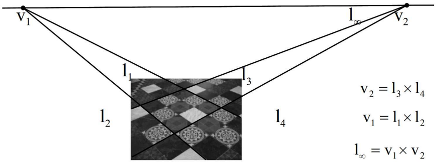
	Euclidean	similarity	affine	projective
Invariants				
length	X			
angle	X	X		
ratio of lengths	X	X		
parallelism	X	X	X	
incidence	X	X	X	X
cross ratio	X	X	X	X

Improper points and the $\mathbf{l}_\infty$

Homography

$$\begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{v}^T & w \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{A} \begin{bmatrix} x \\ y \end{bmatrix} \\ v_1 x + v_2 y \end{bmatrix}$$

- Improper points mapped on finite
- $\mathbf{l}'_\infty = \mathbf{H}^{-T}\mathbf{l}_\infty \neq \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

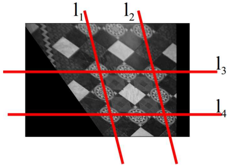


- *Vanishing point*: where world parallel lines converge in image

Affine

$$\begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0} & w \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{A} \begin{bmatrix} x \\ y \end{bmatrix} \\ 0 \end{bmatrix}$$

- Improper points remain at infinity but they change!
- $\mathbf{l}'_\infty = \mathbf{H}^{-T}\mathbf{l}_\infty = \begin{bmatrix} \mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}^T \mathbf{l}_\infty$
  $\mathbf{l}'_\infty = \mathbf{l}_\infty = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$



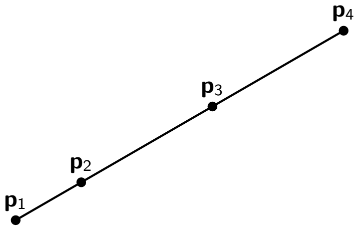
Outline

Cross Ratio

Given

- 4 collinear points $\mathbf{p}_i$
- Distances $d_{ij} = \sqrt{(\mathbf{p}_{i_x} - \mathbf{p}_{j_x})^2 + (\mathbf{p}_{i_y} - \mathbf{p}_{j_y})^2}$

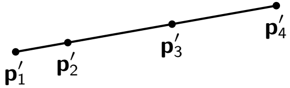
$$CR(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) = \frac{\frac{d_{12}}{d_{13}}}{\frac{d_{24}}{d_{34}}} = \frac{d_{12}\ d_{34}}{d_{13}\ d_{24}}$$



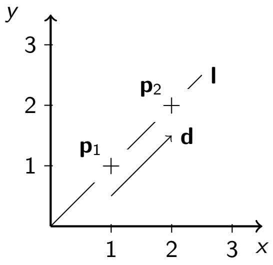
Property

Invariant under any projective transformation

$$CR(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) = CR(\mathbf{p}'_1, \mathbf{p}'_2, \mathbf{p}'_3, \mathbf{p}'_4)$$



Parametric Lines



Line

- $\mathbf{l} = \mathbf{p}_1 \times \mathbf{p}_2$

Direction

- $\mathbf{d}_{12} = \mathbf{p}_2 - \mathbf{p}_1$
  $\mathbf{p}_i$ normalized
- $\mathbf{d}_w = 0$: improper point or direction
- $\overline{\mathbf{d}} = \frac{\mathbf{d}}{\|\mathbf{d}\|}$

Parametric Line

- $\mathbf{p}_\theta = \mathbf{p}_1 + \theta\overline{\mathbf{d}}$
- e.g., $\theta = \|\mathbf{d}\| \rightarrow \mathbf{p}_2$
- e.g., $\theta = 0 \rightarrow \mathbf{p}_1$

Parametric Distance

- Consider $\mathbf{p}_{\theta_1}, \mathbf{p}_{\theta_2}$
- $d_{12} = \|\mathbf{p}_{\theta_2} - \mathbf{p}_{\theta_1}\|$

$$= \|\mathbf{p}_2 + \theta_2\overline{\mathbf{d}} - \mathbf{p}_1 - \theta_1\overline{\mathbf{d}}\|$$
$$= \sqrt{(\theta_2 - \theta_1)^2\,\overline{\mathbf{d}}_x^2 + (\theta_2 - \theta_1)^2\,\overline{\mathbf{d}}_y^2}$$
$$= \sqrt{(\theta_2 - \theta_1)^2\left(\overline{\mathbf{d}}_x^2 + \overline{\mathbf{d}}_y^2\right)}$$
$$= \theta_2 - \theta_1$$

Cross Ratio Example - 1

Image source

Questions

- Identify the vanishing points
- Calculate the $\mathbf{l}'_\infty$
- Identify the vertical middle line
- Identify the field bottom line
- Calculate relative player position
- Identify vanishing point of the diagonal

Cross Ratio Example - 2

VANISHING POINTS - STEP 1

IDENTIFY

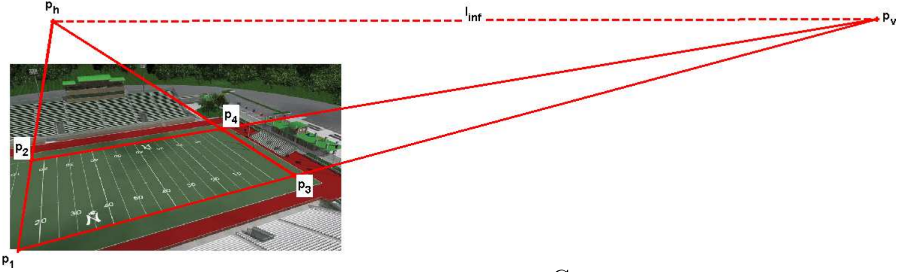
- 4 points on a rectangle
  in the world plane

CALCULATE

- $\mathbf{l}_1 = \mathbf{p}_1 \times \mathbf{p}_2$
- $\mathbf{l}_2 = \mathbf{p}_3 \times \mathbf{p}_4$
- $\mathbf{l}_3 = \mathbf{p}_1 \times \mathbf{p}_3$
- $\mathbf{l}_4 = \mathbf{p}_2 \times \mathbf{p}_4$

Cross Ratio Example - 3

Vanishing points - step 2



Given

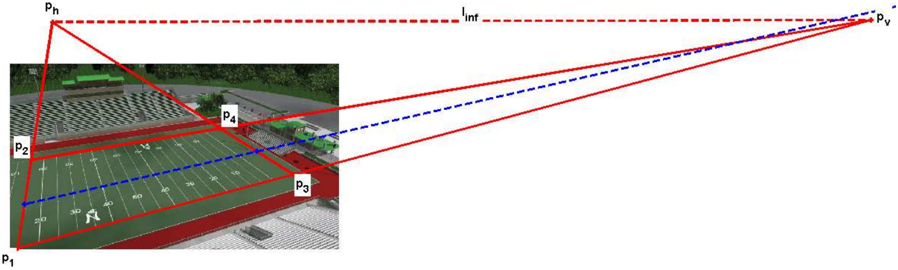
- $\mathbf{l}_1$, $\mathbf{l}_2$, $\mathbf{l}_3$, $\mathbf{l}_4$

Calculate

- $\mathbf{p}_h = \mathbf{l}_1 \times \mathbf{l}_2$
- $\mathbf{p}_v = \mathbf{l}_3 \times \mathbf{l}_4$
- $\mathbf{l}'_\infty = \mathbf{p}_h \times \mathbf{p}_v$

Cross Ratio Example - 4

Vertical middle line - wrong way



Middle point of lines

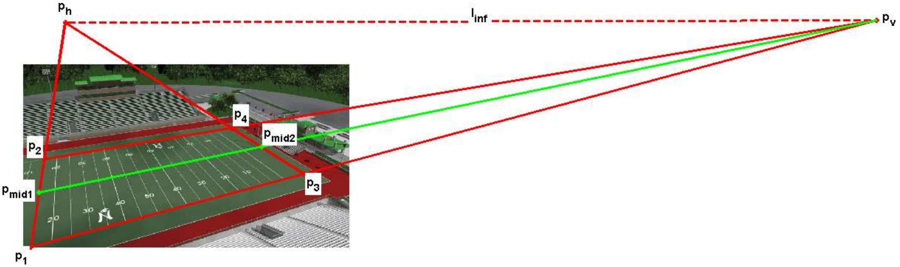
- $\mathbf{p}_{m1} = \frac{1}{2}(\mathbf{p}_1 + \mathbf{p}_2)$
- $\mathbf{p}_{m2} = \frac{1}{2}(\mathbf{p}_3 + \mathbf{p}_4)$
- $\mathbf{l}_m = \mathbf{p}_{m1} \times \mathbf{p}_{m2}$

Wrong

- $\mathbf{l}_m$ has to pass for $\mathbf{p}_v$
  $\rightarrow$ is not the middle line
- Homography doesn't preserve ratios, length, ...

Cross Ratio Example - 5

Vertical middle line - the right way!



In the image

- $CR(\mathbf{p}_1, \mathbf{p}_{m1}, \mathbf{p}_2, \mathbf{p}_h)$ using parametric line

$= CR(0, \theta_m, \theta_2, \theta_h) = \frac{\theta_m(\theta_h - \theta_2)}{\theta_2(\theta_h - \theta_m)}$

Equation

$$\begin{aligned} CR(\mathbf{p}_1, \mathbf{p}_{m1}, \mathbf{p}_2, \mathbf{p}_h) &= CR(0, a, 2a, \infty) \\ \frac{\theta_m(\theta_h - \theta_2)}{\theta_2(\theta_h - \theta_m)} &= 1/2 \end{aligned}$$

In the world

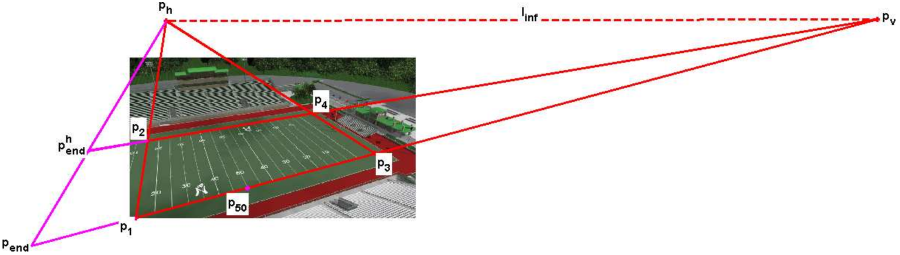
- $CR(0, a, 2a, \infty) = \frac{a \ \infty}{2a \ \infty} = \frac{1}{2}$
- a is the (unknow) half-length

Solution

- $\theta_m = \frac{\theta_2 \theta_h}{2\theta_h - \theta_2}$
- $\mathbf{p}_{m1} = \mathbf{p}_1 + \theta_m \, \overline{\mathbf{d}}_{12}$
- do the same for $\mathbf{p}_{m2}$

Cross Ratio Example - 6

Identify field bottom line



In the image

- Get the p_{50} point (field middle)
- $CR(\mathbf{p}_v, \mathbf{p}_3, \mathbf{p}_{50}, \mathbf{p}_{end}) = CR(0, \theta_3, \theta_m, \theta_{end})$

$$= \frac{\theta_3(\theta_{end} - \theta_m)}{\theta_m(\theta_{end} - \theta_3)}$$

In the world

- $CR(-\infty, 0, a, 2a) = \frac{\infty}{\infty}\frac{a}{2a} = \frac{1}{2}$
- a is the (unknow) half-length

Equation

$$CR(\mathbf{p}_v, \mathbf{p}_3, \mathbf{p}_{50}, \mathbf{p}_{end}) = CR(-\infty, 0, a, 2a)$$

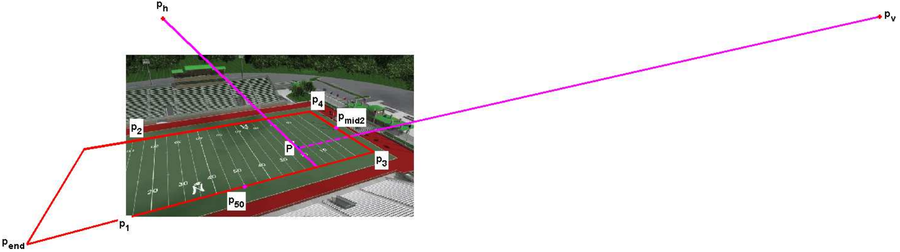
$$\frac{\theta_3(\theta_{end} - \theta_m)}{\theta_m(\theta_{end} - \theta_3)} = 1/2$$

Solution

- $\theta_{end} = \frac{\theta_m \theta_3}{2\theta_3 - \theta_m}$
- $\mathbf{p}_{end} = \mathbf{p}_v + \theta_{end}\bar{\mathbf{d}}_{v3}$
- $\mathbf{l}_{end} = \mathbf{p}_{end} \times \mathbf{p}_h$

Cross Ratio Example - 7

Calculate relative player P position



Origin

- in $\mathbf{p}_3$
- x towards $\mathbf{p}_4$
- y towards $\mathbf{p}_1$

Calculate

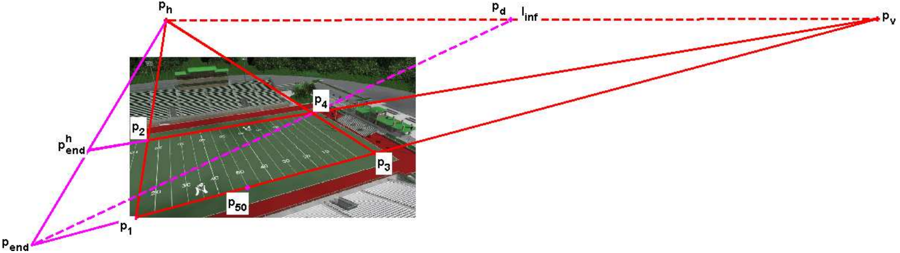
- $\mathbf{P}_x = (\mathbf{P} \times \mathbf{p}_v) \times (\mathbf{p}_3 \times \mathbf{p}_4)$
- $\mathbf{P}_y = (\mathbf{P} \times \mathbf{p}_h) \times (\mathbf{p}_1 \times \mathbf{p}_3)$

Cross Ratio

- $CR(\mathbf{p}_3, \mathbf{P}_x, \mathbf{p}_{mid2}, \mathbf{p}_4) = CR(0, x, \frac{1}{2}, 1)$
  - $\frac{\theta_x(\theta_4 - \theta_{mid2})}{\theta_{mid2}(\theta_4 - \theta_x)} = \frac{x}{1-x}$
- $CR(\mathbf{p}_3, \mathbf{P}_y, \mathbf{p}_{50}, \mathbf{p}_{end}) = CR(0, x, \frac{1}{2}, 1)$
  - $\frac{\theta_y(\theta_{end} - \theta_{50})}{\theta_{50}(\theta_{end} - \theta_y)} = \frac{x}{1-x}$

Cross Ratio Example - 8

Calculate vanishing of the diagonal



Calculate

- $\mathbf{l}_d = \mathbf{p}_{end} \times \mathbf{p}_4$
- $\mathbf{p}_d = \mathbf{l}_d \times \mathbf{l}'_\infty$

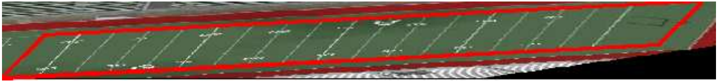
Last step - Affine reconstruction

Affine transformation

- $\mathbf{l}_\infty = [0,\ 0,\ 1]^T$ invariant but not point-wise!
- Consider $\mathbf{l}'_\infty = [l'_x,\ l'_y,\ l'_z]^T$ image of $\mathbf{l}_\infty$
- Consider $\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l'_x & l'_y & l'_z \end{bmatrix}$
- Could be verified that $\mathbf{l}_\infty = \mathbf{H}^{-T}\mathbf{l}'_\infty$
- i.e., $\mathbf{p}_{aff} = \mathbf{H}\,\mathbf{p}_{img}$, $\mathbf{H}$ map points of the image to a affine transformation of the world

Source image

Affine reconstruction



Outline

Projective Geometry - 3D

Points

- Points $\mathbf{p}_e = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \in \mathbb{R}^3$
  in Cartesian coordinates
- $\mathbf{p}_h = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in \mathbb{R}^4$
  in homogeneous coordinates
- $\begin{cases} X & = & x/w \\ Y & = & y/w \\ Z & = & z/w \\ w & \neq & 0 \end{cases}$
- i.e., there is an arbitrary *scale factor*

Planes

- Planes $\pi = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4$
- $\mathbf{n} = \frac{\begin{bmatrix} a, & b, & c \end{bmatrix}^T}{\left\| \begin{bmatrix} a, & b, & c \end{bmatrix}^T \right\|}$
  unitary normal to the plane
- $\mathbf{p}_h \in \pi \Longleftrightarrow \mathbf{p}_h^T \pi = \pi^T \mathbf{p}_h = 0$
- $\pi_\infty = \begin{bmatrix} 0, & 0, & 0, & 1 \end{bmatrix}^T$: plane at infinity
  contains all improper points

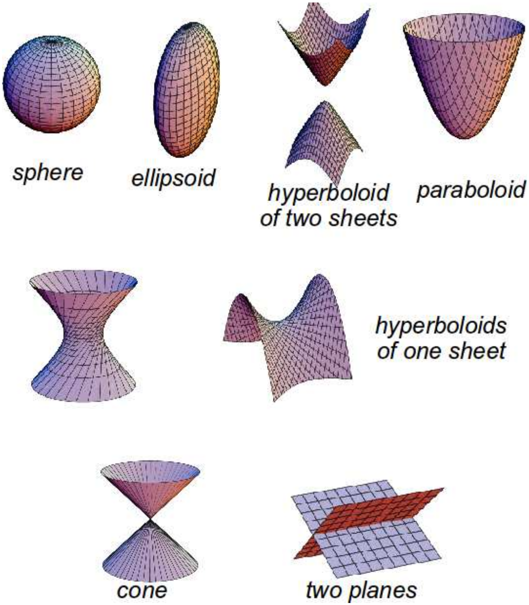
Quadrics

DEFINITION

- Quadratic polynomial equation
- *Quadric surface*
- Matrix form equation
  - $\mathbf{x}^T\mathbf{Q}\mathbf{x} = 0$
- $\mathbf{Q}$ is 4×4 symmetric

$\rightarrow$ $\mathbf{Q}$ is homogeneous too, i.e., 10 parameters, 9 D.O.F.

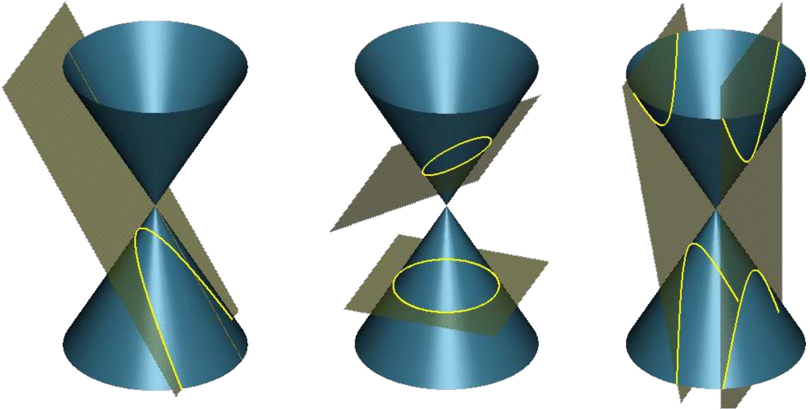
Quadrics - Summmary



Quadrics & conics

INTERSECTION

- $\mathbf{Q} \cap \pi \to$ conic
- Conics are planar sections of quadrics



Hierarchy of transformations

Projective
15dof

$$\begin{bmatrix} A & t \\ v^T & v \end{bmatrix}$$

Affine
12dof

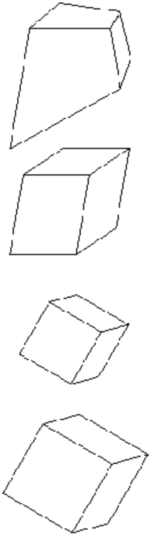
$$\begin{bmatrix} A & t \\ 0^T & 1 \end{bmatrix}$$

Similarity
7dof

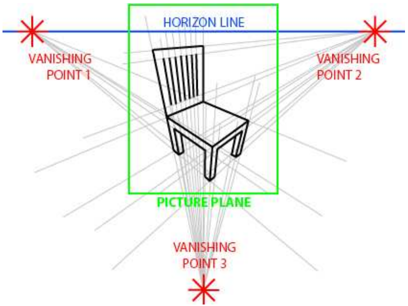
$$\begin{bmatrix} sR & t \\ 0^T & 1 \end{bmatrix}$$

Euclidean
6dof

$$\begin{bmatrix} R & t \\ 0^T & 1 \end{bmatrix}$$



Vanishing points



Vanishing points

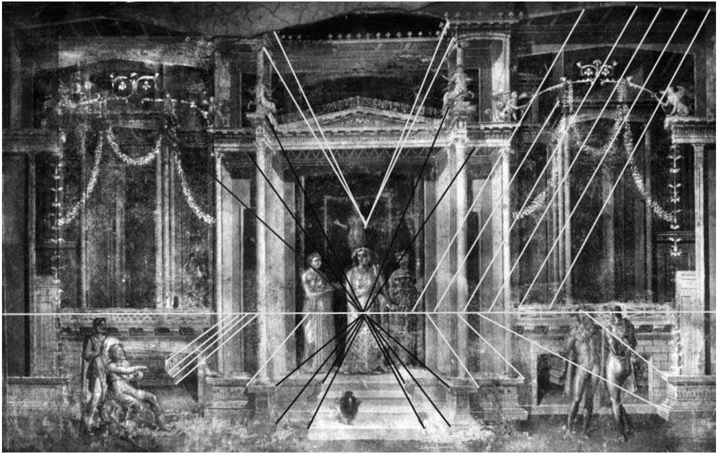
- π_∞ contains all the *directions*
- All the lines with the same direction intersect on π_∞ at the same point
- The vanishing point is the *image* of this intersection

Vanishing lines

- Parallel planes intersect π_∞ in a common line
- The vanishing line is the *image* of this intersection
- e.g., the *horizon line* is the *image* of the intersection of the set of horizontal planes $\{\pi_H\}$ with π_∞

Art & Perspective - 1

Fresco in Pompeii - I B.C.



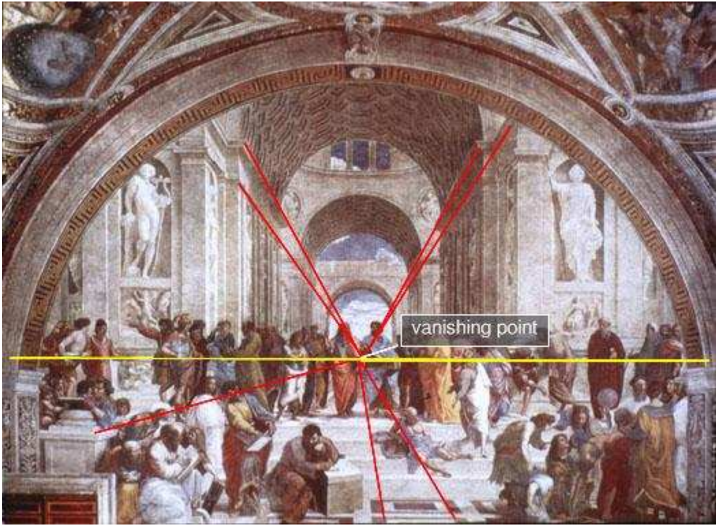
Partially correct perspective

The skill was lost during the middle ages,

it did not reappear in paintings until the Renaissance

Art & Perspective - 2

THE SCHOOL OF ATHENS - RAFFAELLO SANZIO - ~ 1510



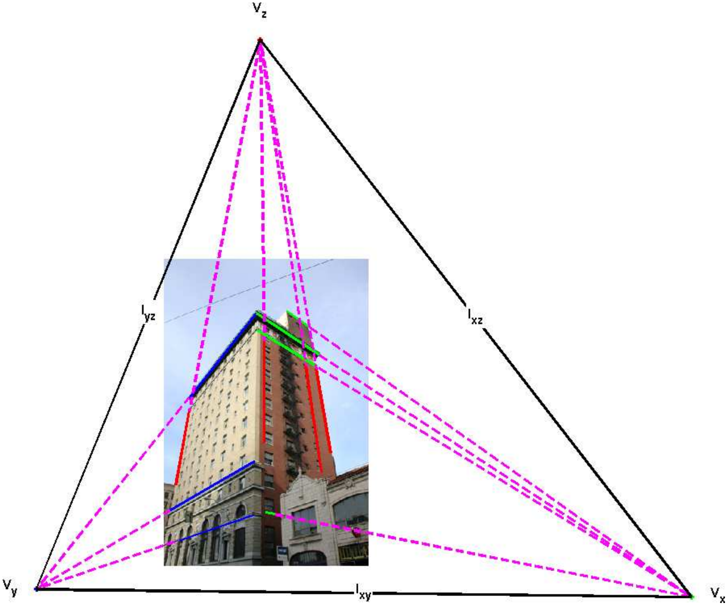
Correct perspective

Vanishing points example - 1

QUESTION

- Find the three vanishing point in the image
- Compute the horizon line in the image
- Compute others vanishing lines $\cdots$

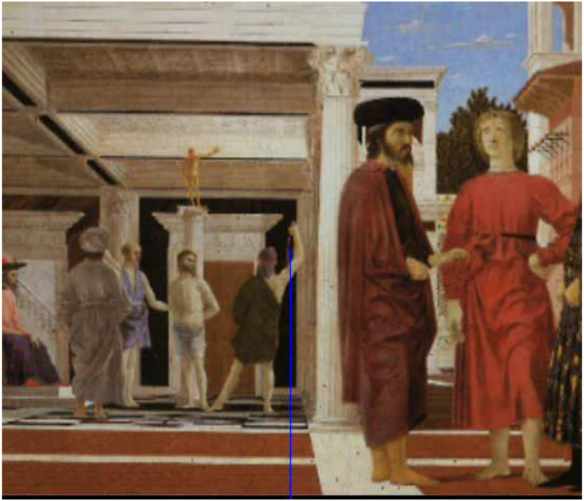
Vanishing points example - 2



Outline

Reconstruction example - 1

Flagellazione di Cristo - Piero della Francesca - ~ 1450



Reconstruction example - 2

TRINITY - MASACCIO - $\sim$ 1426

Reconstruction example - 3

A simple photo

Nice stuff with Projective geometry - 1

Felice Varini - *http://www.varini.org/*

Nice stuff with Projective geometry - 2

Felice Varini - *http://www.varini.org/*

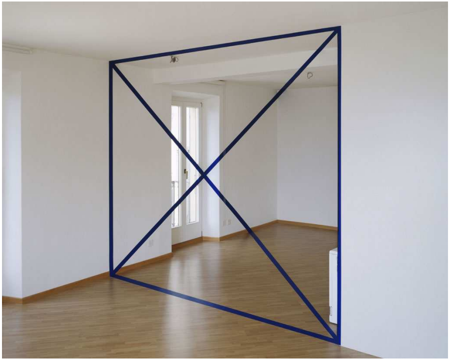
Nice stuff with Projective geometry - 3

Felice Varini - *http://www.varini.org/*

Nice stuff with Projective geometry - 4

Felice Varini - *http://www.varini.org/*

Nice stuff with Projective geometry - 5



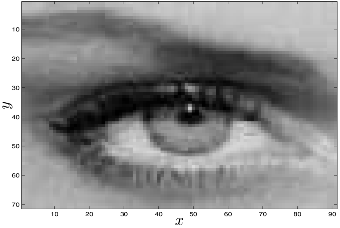
Felice Varini - *http://www.varini.org/*

Outline

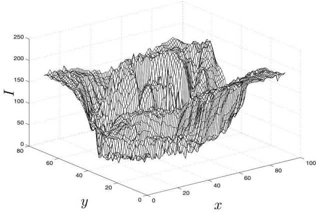
What is an image

Image

- Two-dimensional brightness array: $\mathbf{I}$
- $3\times$ two-dimensional array: $\mathbf{I}_R, \mathbf{I}_G, \mathbf{I}_B$
  - RGB: Red, Green, Blue
  - others: YUV, HSV, HSL, $\cdots$
- Ideal: $\mathbf{I} : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}_+$
- Discrete: $\mathbf{I} : \Omega \subset \mathbb{N}^2 \rightarrow \mathbb{R}_+^*$
  - e.g., $\Omega = [0, 639] \times [0, 479] \subset \mathbb{N}^2$
  - e.g., $\Omega = [1, 1024] \times [1, 768] \subset \mathbb{N}^2$
  - e.g., $\mathbb{R}_+^* = [0, 255] \subset \mathbb{N}$
  - e.g., $\mathbb{R}_+^* = [0, 1] \subset \mathbb{R}$
- $\mathbf{I}(x, y)$ is the intensity
- $\mathbf{I}$ result of 3D $\rightarrow$ 2D projection: *flat*



188	186	188	187	168	130	101	99	110	113	112	107	117	140	153	153	156	158	156	153
189	189	188	181	163	135	109	104	113	113	110	109	117	134	147	152	156	163	160	156
190	190	188	176	159	139	115	106	114	123	114	111	119	130	141	154	165	160	156	151
190	188	188	175	158	139	114	103	113	126	112	113	127	133	137	151	165	156	152	145
191	185	189	177	158	138	110	99	112	119	107	115	137	140	135	144	157	163	158	150
193	183	178	164	148	134	118	112	119	117	118	106	122	139	140	152	154	160	155	147
185	181	178	165	149	135	121	116	124	120	122	109	123	139	141	154	156	159	154	147
175	176	176	163	145	131	120	118	125	123	125	112	124	139	142	155	158	158	155	148
170	170	172	159	137	123	116	114	119	122	126	113	123	137	141	156	158	159	157	150
171	171	173	157	131	119	116	113	114	118	125	113	122	135	140	155	156	160	160	152
174	175	176	156	128	120	121	118	113	112	123	114	122	135	141	155	155	158	159	152
176	174	174	151	123	119	126	121	112	108	122	115	123	137	143	156	155	152	155	150
175	169	168	144	117	117	127	122	109	106	122	116	125	139	145	158	156	147	152	148
179	179	180	155	127	121	118	109	107	113	125	133	130	129	139	153	161	148	155	157
176	183	181	153	122	115	113	106	105	109	123	132	131	131	140	151	157	149	156	159
180	181	177	147	115	110	111	107	107	105	120	132	133	133	141	150	154	148	155	157
181	174	170	141	113	111	115	112	113	105	119	130	132	134	144	153	156	148	152	151
180	172	168	140	114	114	118	113	112	107	119	128	130	134	146	157	162	153	153	148
186	176	171	142	114	114	116	110	108	104	116	125	128	134	148	161	165	159	157	149
185	178	171	138	109	110	114	110	109	97	110	121	127	136	150	160	163	158	156	150



Camera

Optical system

- Set of lenses to direct light

  change in the direction of propagation

- CCD sensor

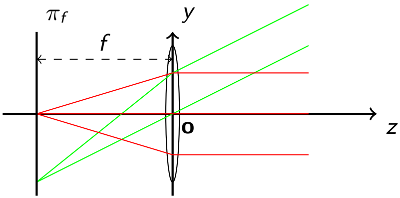
  integrate energy both

  - in time (exposure time)
  - in space (pixel size)

Thin lenses model

Thin lenses

- Mathematical model
  - Optical axis (z)
  - Focal plane π_f ($\perp z$)
  - Optical center **o**
- Parameters
  - f distance **o**, π_f
- Property
  - Parallel rays converge π_f
  - Rays through **o** undeflected



Rays from scene

Image from a scene point **P**

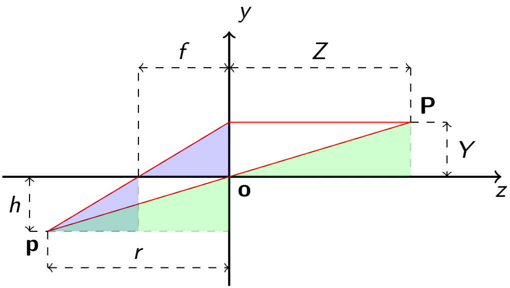
- $\mathbf{P} = (Z, Y)$
- Ray through **o** undeflected
- Ray parallel to z cross in $(-f, 0)$

Similarities

- Blue triangles: $\dfrac{h}{Y} = \dfrac{r - f}{f}$
- Green triangles: $\dfrac{h}{Y} = \dfrac{r}{Z}$

Fresnel Law

- $\dfrac{1}{Z} + \dfrac{1}{r} = \dfrac{1}{f}$
- Note: $Z \to \infty \Rightarrow r \to f$



The image plane

Image plane $\pi_{\mathbf{I}}$

- Plane $\perp$ z at distance d

Blur Circle

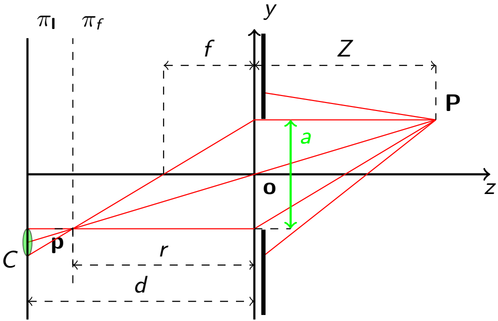
- If $d \neq r$

  image of **P** is a circle C

- Diameter of C:

$$\phi(C) = \frac{a(d-r)}{r}$$

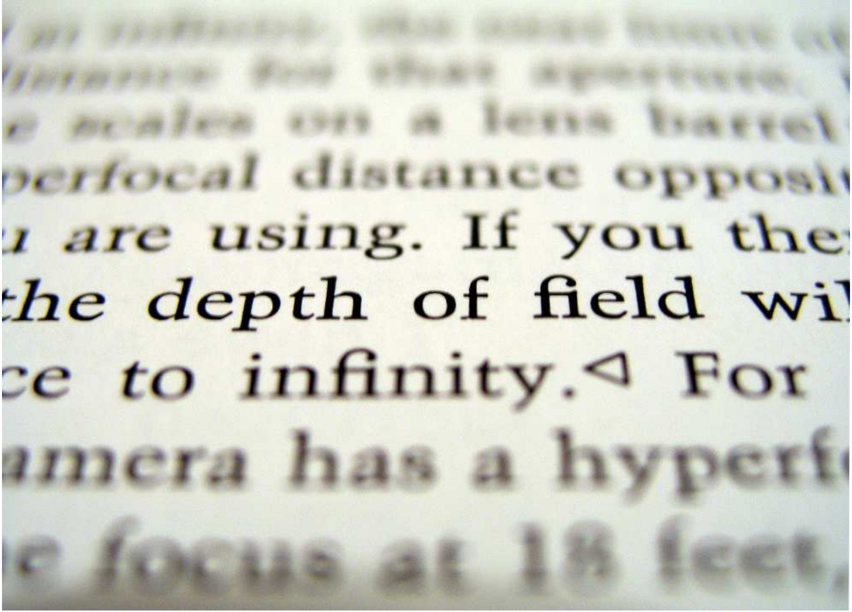
  a is the *aperture*



Focused image

- $\phi(C) <$ pixel size
- Depth of field : range $[Z_1, Z_2]$: $\phi(C) <$ pixel size

Depth of field - Example 1



Depth of field - Example 2

The same scene - different aperture

Outline

Pin hole model - Definition

HYPOTHESIS

- $Z \gg a$
- $Z \gg f \rightarrow r \sim f$

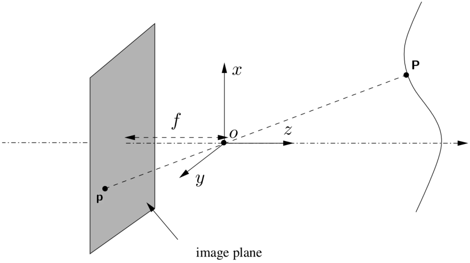
IMAGE OF $\mathbf{P}$

- $\mathbf{l}_{Po}$: line that join $\mathbf{P}$ and $\mathbf{o}$
- $\mathbf{p} = \pi_{\mathbf{I}} \cap \mathbf{l}_{Po}$

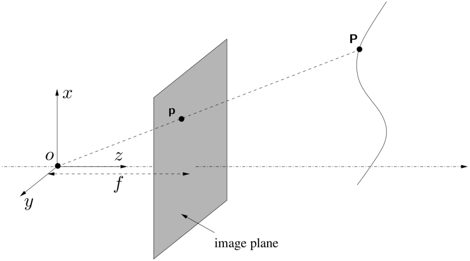
NOTES

- $\mathbf{p}$ is the image of $\forall \mathbf{P}_i \in \mathbf{l}_{Po}$
- $\mathbf{l}_{Po}$: interpretation line of $\mathbf{p}$

PIN-HOLE MODEL



FRONTAL PIN-HOLE MODEL



Pin hole model - Geometry

Given

- $\mathbf{P}^{(O)} = \left[X, Y, Z, 1\right]^T$
- $\mathbf{p}^{(\mathbf{I})} = \left[x, y, 1\right]^T$

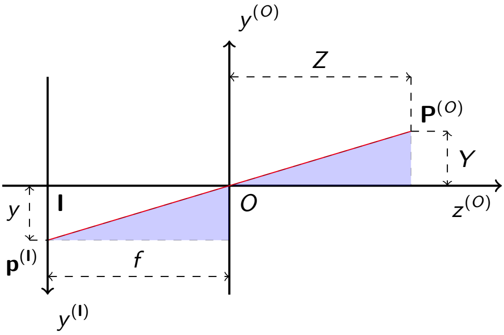
Projection

- $y = f\dfrac{Y}{Z}$
- $x = f\dfrac{X}{Z}$

  look at the triangles

Note

- $\lambda\mathbf{P}^{(O)}$ projects on $\mathbf{p}^{(\mathbf{I})}$
- $\left[sX, sY, sZ, 1\right]^T$ projects on $\mathbf{p}^{(\mathbf{I})}$, $\forall s \neq 0$



Pin hole model - Matrix

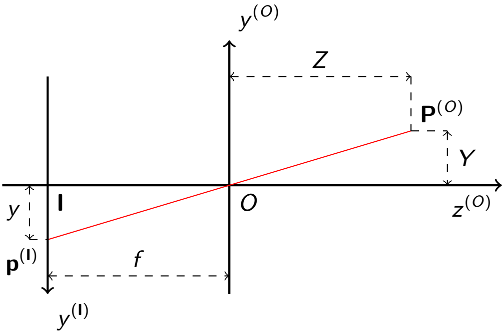
Projection equations

- $y = f\dfrac{Y}{Z}$
- $x = f\dfrac{X}{Z}$

In matrix form

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix}$$

$$\mathbf{p}^{(\mathbf{I})} = \boldsymbol{\pi}\ \mathbf{P}^{(O)}$$



Define

- $\mathbf{K} = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix}$: intrinsic parameters
- $\boldsymbol{\pi} = \begin{bmatrix} \mathbf{K} & \mathbf{0} \end{bmatrix}$: projection matrix

Pin hole model - Image coordinates - 1

Reference system on image

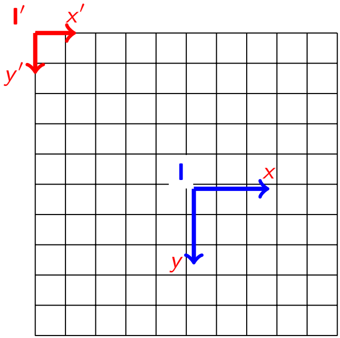
- $\mathbf{I}$: origin centered on $z^{(O)} \cap \pi_{\mathbf{I}}$
- $\mathbf{I}'$: origin centered top-left image
- $\mathbf{c}^{(\mathbf{I}')} = \left[\mathbf{c}_x, \, \mathbf{c}_y\right]^T$: position of $\mathbf{I}$ in $\mathbf{I}'$

Metric

- $\mathbf{I}$ metric
- $\mathbf{I}'$ in pixel
- $\mathbf{c}^{(\mathbf{I}')}$ in pixel

Definition

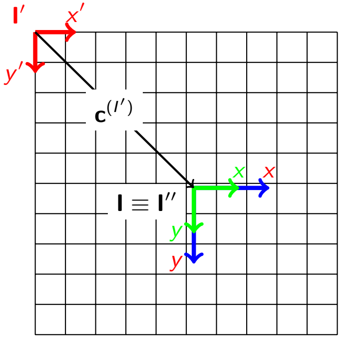
- $\left[0, 0\right]^{T(\mathbf{I})} \equiv \left[\mathbf{c}_x, \, \mathbf{c}_y\right]^{T(\mathbf{I}')}$: *principal point*
- Image of the optical center ($\mathbf{o}$) *or* $z^{(O)} \cap \pi_{\mathbf{I}}$



Pin hole model - Image coordinates - 2

Meters to pixels

- Consider $\mathbf{I}''$: origin on $\mathbf{I}$, *in pixel*
- Scale meters to pixels
  - $\mathbf{p}_x^{(\mathbf{I}'')} = \mathbf{s}_x \mathbf{p}_x^{(\mathbf{I})}$
  - $\mathbf{p}_y^{(\mathbf{I}'')} = \mathbf{s}_y \mathbf{p}_y^{(\mathbf{I})}$
- $\mathbf{s}_x = \frac{1}{d_x}$, d_x: width of a pixel $[m]$
- $\mathbf{s}_y = \frac{1}{d_y}$, d_y: height of a pixel $[m]$
- $\mathbf{s}_x = \mathbf{s}_y$: square pixel
- $$\mathbf{p}^{(\mathbf{I}'')} = \begin{bmatrix} \mathbf{s}_x & 0 & 0 \\ 0 & \mathbf{s}_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{p}^{(\mathbf{I})}$$



Translation

- $$\mathbf{p}^{(\mathbf{I}')} = \begin{bmatrix} 1 & 0 & \mathbf{c}_x \\ 0 & 1 & \mathbf{c}_y \\ 0 & 0 & 1 \end{bmatrix} \mathbf{p}^{(\mathbf{I}'')}$$

Pin hole model - Intrinsic camera matrix

Consider

- $$\mathbf{p}^{(\mathbf{I})} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{P}^{(O)}$$
- $$\mathbf{p}^{(\mathbf{I}'')} = \begin{bmatrix} \mathbf{s}_x & 0 & 0 \\ 0 & \mathbf{s}_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{p}^{(\mathbf{I})}$$
- $$\mathbf{p}^{(\mathbf{I}')} = \begin{bmatrix} 1 & 0 & \mathbf{c}_x \\ 0 & 1 & \mathbf{c}_y \\ 0 & 0 & 1 \end{bmatrix} \mathbf{p}^{(\mathbf{I}'')}$$

In one step

- $$\mathbf{p}^{(\mathbf{I}')} = \begin{bmatrix} \mathbf{s}_x f & 0 & \mathbf{c}_x & 0 \\ 0 & \mathbf{s}_y f & \mathbf{c}_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{P}^{(O)}$$

The intrinsic camera matrix

or *calibration matrix*

$$\mathbf{K} = \begin{bmatrix} f_x & s & \mathbf{c}_x \\ 0 & f_y & \mathbf{c}_y \\ 0 & 0 & 1 \end{bmatrix}$$

- f_x, f_y: focal lenght (in pixels)
  $f_x/f_y = s_x/s_y = a$: aspect ratio
- s: skew factor
  pixel not orthogonal
  usually 0 in modern cameras
- $\mathbf{c}_x$, $\mathbf{c}_y$: principal point (in pixel)
  usually $\neq$ half image size due to misalignment of CCD

Outline

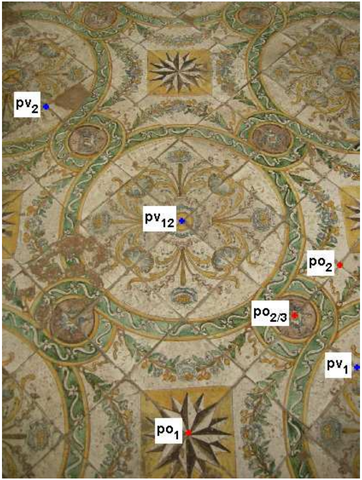
Exercise 1 - Tiles

IMAGE SOURCE

QUESTIONS

- Identify the vanishing points
- using cross ratio
- i.e., without use parallel lines

Exercise 1 - Tiles - Solution



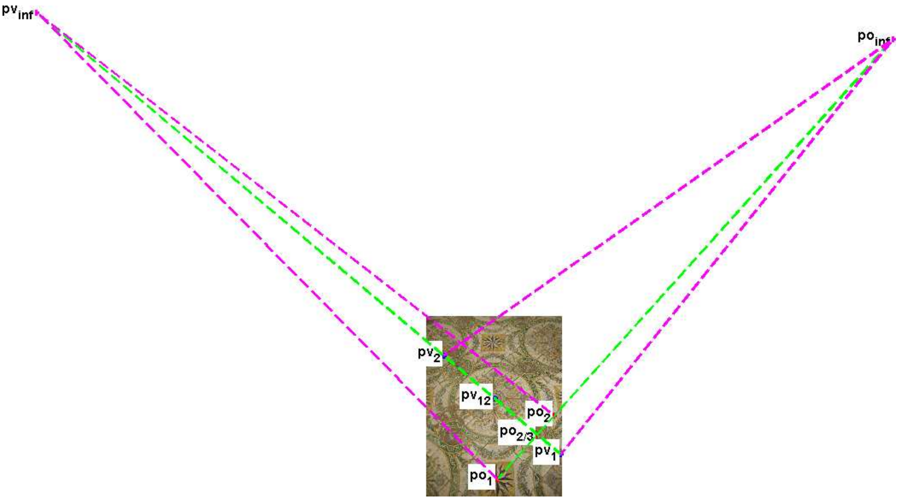
HORIZONTAL

- $CR(\mathbf{p}_{o1}, \mathbf{p}_{o23}, \mathbf{p}_{o2}, \mathbf{p}_{o\infty}) = CR(0, a2/3, a, \infty)$
- $CR(0, \theta_{23}, \theta_3, \theta_o) = 2/3$
- $\theta_o = \frac{-\theta_{23}\theta_3}{2\theta_3 - 3\theta_2 3}$
- $\mathbf{p}_{o\infty} = \mathbf{p}_{o1} + \theta_o \bar{\mathbf{d}}_o$

VERTICAL

- $CR(\mathbf{p}_{v1}, \mathbf{p}_{v12}, \mathbf{p}_{v2}, \mathbf{p}_{v\infty}) = CR(0, a, 2a, \infty)$
- $CR(0, \theta_{12}, \theta_{12}, \theta_v) = 1/2$
- $\theta_v = \frac{\theta_{12}(\theta_v - \theta_2)}{\theta_2(\theta_v - \theta_{12})}$
- $\mathbf{p}_{v\infty} = \mathbf{p}_{v1} + \theta_v \bar{\mathbf{d}}_v$

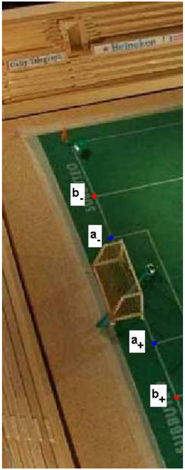
Exercise 1 - Tiles - Check



Magenta lines only for check correctness

Exercise 2 - Soccer field

Image source



Find

- Center of the goal-line
- Vanishing point of the goal-line

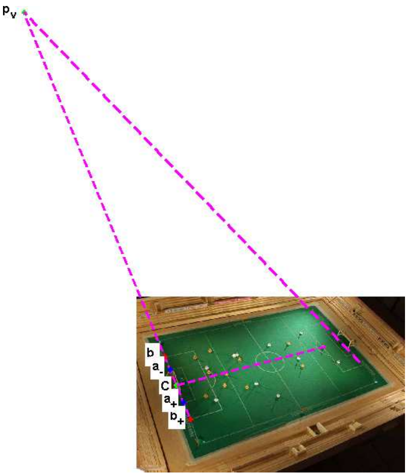
Solution

- 4 symmetric points
- a_-, a_+ and b_-, b_+

$$\begin{cases} CR(0, -a, a, \infty) & = & CR(\theta_c, \theta_{a_-}, \theta_{a_+}, \theta_v) \\ CR(0, -b, b, \infty) & = & CR(\theta_c, \theta_{b_-}, \theta_{b_+}, \theta_v) \end{cases}$$

- 2 equations, 2 unknown
- 4 solutions, only 2 are are valid

Exercise 2 - Soccer field



Magenta lines only for check correctness