Robotics - Projective Geometry and Camera model

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Outline

Outline

Projective Transformations - Recall

Projective Transformation

$$\mathbf{x}' = \mathbf{H}\,\mathbf{x}$$

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

Projective Transformations - Recall

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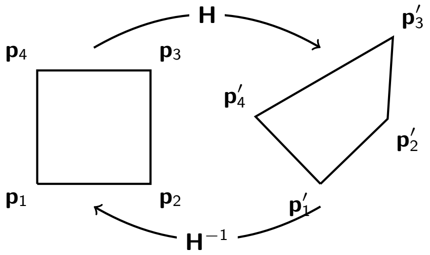
Notes

- Map plane to plane
- It's a linear transformation

  in homogeneous coordinates
- It's homogeneous too $\lambda\mathbf{H} \equiv \mathbf{H}$



Projective Transformations - Image Rectification - 1

Homography Estimation

- Take four point on first image $\mathbf{x}_i = \left[x_i,\ y_i,\ w_i\right]^T$
- Map on four known destination points $\mathbf{x}'_i = \left[x'_i,\ y'_i\right]^T$
- Rewrite: $\begin{cases} x''_i &= h_{11}x_i + h_{12}y_i + h_{13}w_i \\ y''_i &= h_{21}x_i + h_{22}y_i + h_{23}w_i \\ w''_i &= h_{31}x_i + h_{32}y_i + h_{33}w_i \end{cases}$
- In cartesian: $\begin{cases} x'_i &= \frac{h_{11}x_i + h_{12}y_i + h_{13}w_i}{h_{31}x_i + h_{32}y_i + h_{33}w_i} \\ y'_i &= \frac{h_{21}x_i + h_{22}y_i + h_{23}w_i}{h_{31}x_i + h_{32}y_i + h_{33}w_i} \end{cases}$
- Fix $h_{33} = 1$ and rewrite $\begin{cases} x'_i(h_{31}x_i + h_{32}y_i + w_i) &= h_{11}x_i + h_{12}y_i + h_{13}w_i \\ y'_i(h_{31}x_i + h_{32}y_i + w_i) &= h_{21}x_i + h_{22}y_i + h_{23}w_i \end{cases}$

Projective Transformations - Image Rectification - 2

- Expand and separate $\begin{cases} x_i h_{11} + y_i h_{12} + w_i h_{13} - x_i' x_i h_{31} - x_i' y_i h_{32} & = & x_i' w_i \\ x_i h_{21} + y_i h_{22} + w_i h_{23} - y_i' x_i h_{31} - y_i' y_i h_{32} & = & y_i' w_i \end{cases}$
- Matrix form (2-lines for each point)

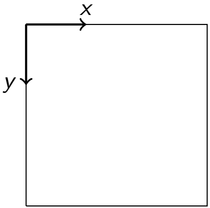
$$\begin{bmatrix} x_1 & y_1 & w_1 & 0 & 0 & 0 & -x_1'x_1 & -x_1'y_1 \\ 0 & 0 & 0 & x_1 & y_1 & w_1 & -x_1'x_1 & -x_1'y_1 \\ x_2 & y_2 & w_2 & 0 & 0 & 0 & -x_2'x_2 & -x_2'y_2 \\ 0 & 0 & 0 & x_2 & y_2 & w_2 & -x_2'x_2 & -x_2'y_2 \\ x_3 & y_3 & w_3 & 0 & 0 & 0 & -x_3'x_3 & -x_3'y_3 \\ 0 & 0 & 0 & x_3 & y_3 & w_3 & -x_3'x_3 & -x_3'y_3 \\ x_4 & y_4 & w_4 & 0 & 0 & 0 & -x_4'x_4 & -x_4'y_4 \\ 0 & 0 & 0 & x_4 & y_4 & w_4 & -x_4'x_4 & -x_4'y_4 \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \end{bmatrix} = \begin{bmatrix} x_1'w_1 \\ y_1'w_1 \\ x_2'w_2 \\ y_2'w_2 \\ x_3'w_3 \\ y_3'w_3 \\ x_4'w_4 \\ y_4'w_4 \end{bmatrix}$$

- System $\mathbf{Ax} = \mathbf{b}$ e.g. in Matlab solved with $\mathbf{x} = \mathbf{A} \backslash \mathbf{b}$

Projective Transformations - Image Rectification - Example

Original Image

Image reference system



- $\mathbf{x}$: {179,525}, {187,73}, {690,307}, {698,467}
- $\mathbf{x}'$: {0,180}, {0,0}, {822,0}, {822,180}
- $$\mathbf{H} = \begin{bmatrix} 0.4659 & 0.0082 & -87.7293 \\ -0.1573 & 0.3382 & 4.7322 \\ -0.0011 & 0.0001 & 1.0000 \end{bmatrix}$$

Projective Transformations - Lines and Conics

Points

$$\mathbf{p}' = \mathbf{H}\mathbf{p}$$

Lines

$$\mathbf{l}' = \mathbf{H}^{-T}\mathbf{l}$$

Proof

- $\mathbf{l}^T\mathbf{p} = 0$
- $\mathbf{l}'^T\mathbf{p}' = 0$
- $\mathbf{l}'^T\mathbf{H}\mathbf{p} = 0$
- $(\mathbf{H}^{-T}\mathbf{l})^T\mathbf{H}\mathbf{p} = 0$
- $\mathbf{l}^T\mathbf{H}^{-1}\mathbf{H}\mathbf{p} = 0$

Conics

$$\mathbf{C}' = \mathbf{H}^{-T}\mathbf{C}\mathbf{H}^{-1}$$

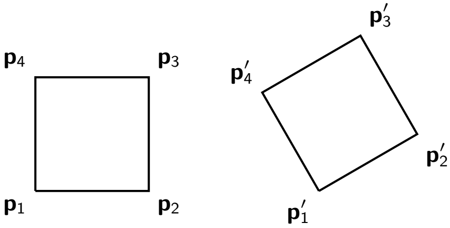
Proof

- $\mathbf{p}^T\mathbf{C}\mathbf{p} = 0$
- $\mathbf{p}'^T\mathbf{C}'\mathbf{p}' = 0$
- $(\mathbf{H}\mathbf{p})^T\mathbf{H}^{-T}\mathbf{C}\mathbf{H}^{-1}\mathbf{H}\mathbf{p} = 0$

Outline

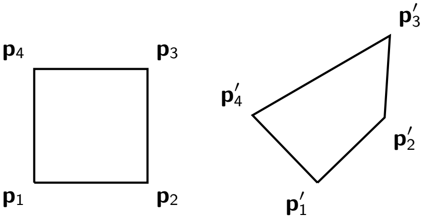
Transformations - Recall

Rototranslation



$$\mathbf{p}' = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \mathbf{p}'$$

Homography

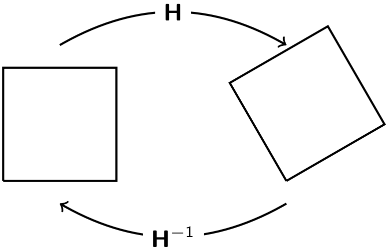


$$\mathbf{p}' = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \mathbf{p}'$$

Class I - Isometries - i.e., Rototranslations

$$\mathbf{p}' = \begin{bmatrix} \xi\cos(\theta) & -\sin(\theta) & \mathbf{t}_x \\ \xi\sin(\theta) & \cos(\theta) & \mathbf{t}_y \\ 0 & 0 & 1 \end{bmatrix} \mathbf{p}'$$

- *iso*: same, *metric*: measure
- $\xi = +1$ orientation preserving
- $\xi = -1$ orientation reversing
- 3 DoF (2 translation, 1 rotation)
- Special cases:
  - Pure rotation
  - Pure translation



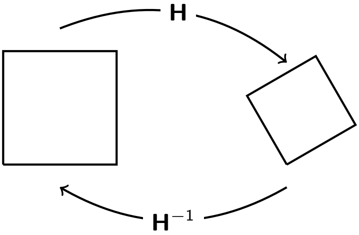
<u>INVARIANTS</u>

- Length
- Area
- Angle

Class II - Similarities

$$\mathbf{p}' = \begin{bmatrix} s\cos(\theta) & -s\sin(\theta) & \mathbf{t}_x \\ s\sin(\theta) & s\cos(\theta) & \mathbf{t}_y \\ 0 & 0 & 1 \end{bmatrix} \mathbf{p}'$$

- Isometry + scale factor
- 4 DoF (2 translation, 1 rotation, 1 scale)
- $det(s\mathbf{R}) = s$



INVARIANTS

- Shape
- Ratios of length
- Ratios of areas
- Angle
- Parallel lines

Class III - Affine transformations

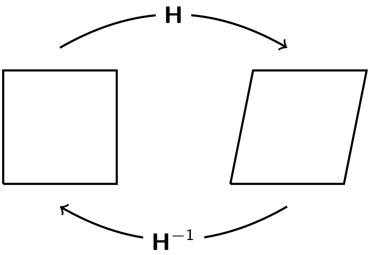
$$\mathbf{p}' = \begin{bmatrix} a_{11} & a_{11} & \mathbf{t}_x \\ a_{21} & a_{22} & \mathbf{t}_y \\ 0 & 0 & 1 \end{bmatrix} \mathbf{p}'$$

- Non-isotropic scaling
- 6 DoF

  (2 translation, 2 rotation, 2 scale)
- $\mathbf{A} = \begin{bmatrix} a_{11} & a_{11} \\ a_{21} & a_{22} \end{bmatrix} = \mathbf{UDV}^T$
- $\mathbf{UDV}^T = \left(\mathbf{UV}^T\right)\left(\mathbf{VDV}^T\right)$

  $\mathbf{U}$, $\mathbf{V}$ orthogonal, $\mathbf{D}$ diagonal
- $\mathbf{R}(\theta)\left(\mathbf{R}(-\phi)\mathbf{D}\mathbf{R}(\phi)\right)$

  rotation on scaled axis



INVARIANTS

- Parallel lines
- Ratios of parallel segment lengths
- Ratios of areas

Class IV - Homographies

$$\mathbf{p}' = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \mathbf{p}'$$

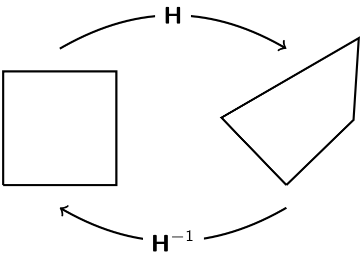
- Mapping plane to plane

  linear in homogeneous coordinates

- 8 DoF

  2 translation, 2 rotation,

  2 scale, 2 for $\mathbf{l}\infty$



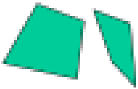
INVARIANTS

- Collinearities
- Cross-ratio of four points on a line

2D Transformations overview

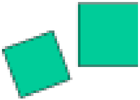
Projective
8dof

$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$



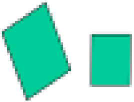
Similarity
4dof

$$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



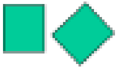
Affine
6dof

$$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



Euclidean
3dof

$$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



	Euclidean	similarity	affine	projective
Transformations				
rotation	X	X	X	X
translation	X	X	X	X
uniform scaling		X	X	X
nonuniform scaling			X	X
shear			X	X
perspective projection				X
composition of projections				X

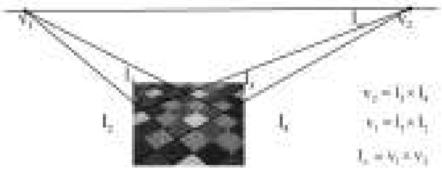
	Euclidean	similarity	affine	projective
Invariants				
length	X			
angle	X	X		
ratio of lengths	X	X		
parallelism	X	X	X	
incidence	X	X	X	X
cross ratio	X	X	X	X

Improper points and the $\mathbf{l}_\infty$

Homography

$$\begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{v}^T & w \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{A} \begin{bmatrix} x \\ y \end{bmatrix} \\ v_1 x + v_2 y \end{bmatrix}$$

- Improper points mapped on finite
- $\mathbf{l}'_\infty = \mathbf{H}^{-T} \mathbf{l}_\infty \neq \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$



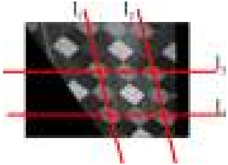
- *Vanishing point*: where world parallel lines converge in image

Affine

$$\begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0} & w \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{A} \begin{bmatrix} x \\ y \end{bmatrix} \\ 0 \end{bmatrix}$$

- Improper points remain at infinity but they change!
- $\mathbf{l}'_\infty = \mathbf{H}^{-T} \mathbf{l}_\infty = \begin{bmatrix} \mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}^T \mathbf{l}_\infty$

  $\mathbf{l}'_\infty = \mathbf{l}_\infty = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$



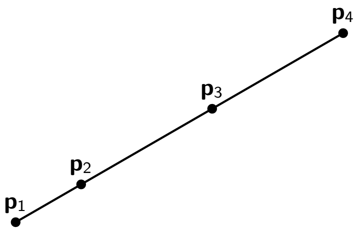
Outline

Cross Ratio

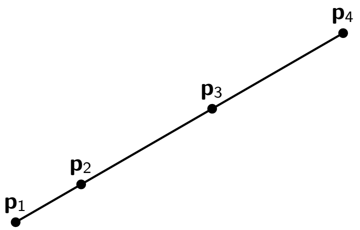
<u>GIVEN</u>

- 4 collinear points $\mathbf{p}_i$
- Distances $d_{ij} = \sqrt{(\mathbf{p}_{i_x} - \mathbf{p}_{j_x})^2 + (\mathbf{p}_{i_y} - \mathbf{p}_{j_y})^2}$

$$CR(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) = \frac{\frac{d_{12}}{d_{13}}}{\frac{d_{24}}{d_{34}}} = \frac{d_{12}}{d_{13}} \frac{d_{34}}{d_{24}}$$



Cross Ratio



GIVEN

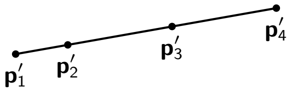
- 4 collinear points $\mathbf{p}_i$
- Distances $d_{ij} = \sqrt{(\mathbf{p}_{i\,x} - \mathbf{p}_{j\,x})^2 + (\mathbf{p}_{i\,y} - \mathbf{p}_{j\,y})^2}$

$$CR(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) = \frac{\frac{d_{12}}{d_{13}}}{\frac{d_{24}}{d_{34}}} = \frac{d_{12}}{d_{13}}\frac{d_{34}}{d_{24}}$$

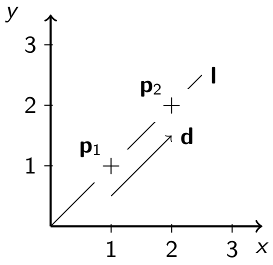
PROPERTY

Invariant under any projective transformation

$$CR(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) = CR(\mathbf{p}'_1, \mathbf{p}'_2, \mathbf{p}'_3, \mathbf{p}'_4)$$



Parametric Lines



Line

- $\mathbf{l} = \mathbf{p}_1 \times \mathbf{p}_2$

Direction

- $\mathbf{d}_{12} = \mathbf{p}_2 - \mathbf{p}_1$
  $\mathbf{p}_i$ normalized
- $\mathbf{d}_w = 0$: improper point or direction
- $\overline{\mathbf{d}} = \frac{\mathbf{d}}{\|\mathbf{d}\|}$

Parametric Line

- $\mathbf{p}_\theta = \mathbf{p}_1 + \theta\overline{\mathbf{d}}$
- e.g., $\theta = \|\mathbf{d}\| \rightarrow \mathbf{p}_2$
- e.g., $\theta = 0 \rightarrow \mathbf{p}_1$

Parametric Distance

- Consider $\mathbf{p}_{\theta_1}$, $\mathbf{p}_{\theta_2}$
- $d_{12} = \|\mathbf{p}_{\theta_2} - \mathbf{p}_{\theta_1}\|$

$$= \|\mathbf{p}_2 + \theta_2\overline{\mathbf{d}} - \mathbf{p}_1 - \theta_1\overline{\mathbf{d}}\|$$
$$= \sqrt{(\theta_2 - \theta_1)^2\,\overline{\mathbf{d}}_x^2 + (\theta_2 - \theta_1)^2\,\overline{\mathbf{d}}_y^2}$$
$$= \sqrt{(\theta_2 - \theta_1)^2\left(\overline{\mathbf{d}}_x^2 + \overline{\mathbf{d}}_y^2\right)}$$
$$= \theta_2 - \theta_1$$

Cross Ratio Example - 1

Image source

Questions

- Identify the vanishing points
- Calculate the $\mathbf{l}'_\infty$
- Identify the vertical middle line
- Identify the field bottom line
- Calculate relative player position
- Identify vanishing point of the diagonal

Cross Ratio Example - 2

Vanishing points - step 1

Identify

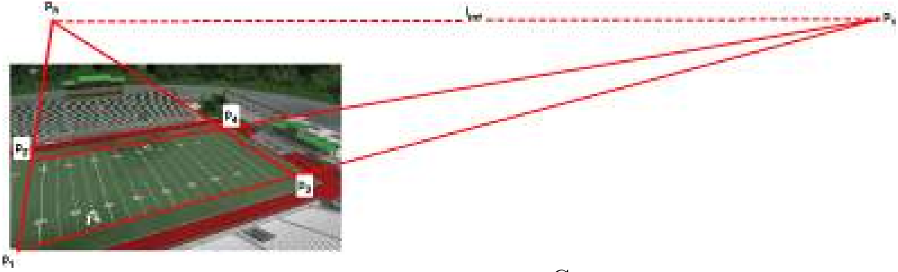
- 4 points on a rectangle
  in the world plane

Calculate

- $\mathbf{l}_1 = \mathbf{p}_1 \times \mathbf{p}_2$
- $\mathbf{l}_2 = \mathbf{p}_3 \times \mathbf{p}_4$
- $\mathbf{l}_3 = \mathbf{p}_1 \times \mathbf{p}_3$
- $\mathbf{l}_4 = \mathbf{p}_2 \times \mathbf{p}_4$

Cross Ratio Example - 3

Vanishing points - step 2



Given

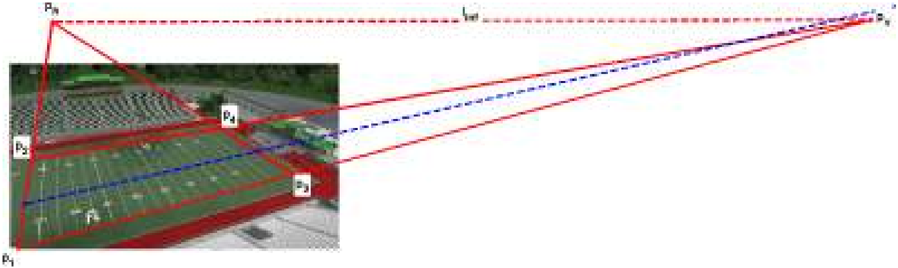
- $\mathbf{l}_1$, $\mathbf{l}_2$, $\mathbf{l}_3$, $\mathbf{l}_4$

Calculate

- $\mathbf{p}_h = \mathbf{l}_1 \times \mathbf{l}_2$
- $\mathbf{p}_v = \mathbf{l}_3 \times \mathbf{l}_4$
- $\mathbf{l}'_\infty = \mathbf{p}_h \times \mathbf{p}_v$

Cross Ratio Example - 4

Vertical middle line - wrong way



Middle point of lines

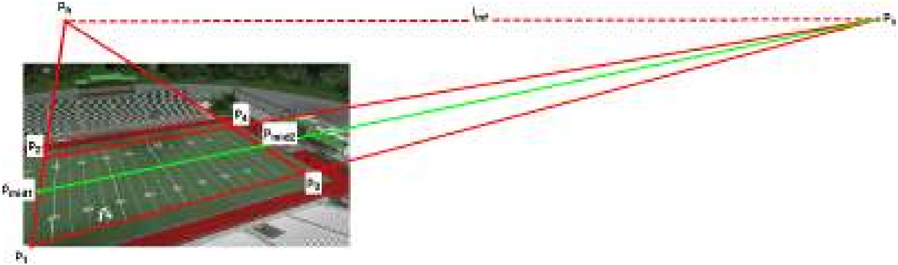
- $\mathbf{p}_{m1} = \frac{1}{2}(\mathbf{p}_1 + \mathbf{p}_2)$
- $\mathbf{p}_{m2} = \frac{1}{2}(\mathbf{p}_3 + \mathbf{p}_4)$
- $\mathbf{l}_m = \mathbf{p}_{m1} \times \mathbf{p}_{m2}$

Wrong

- $\mathbf{l}_m$ has to pass for $\mathbf{p}_v$
  $\rightarrow$ is not the middle line
- Homography doesn't preserve ratios, length, ...

Cross Ratio Example - 5

Vertical middle line - the right way!



In the image

- $CR(\mathbf{p}_1, \mathbf{p}_{m1}, \mathbf{p}_2, \mathbf{p}_h)$ using parametric line

$= CR(0, \theta_m, \theta_2, \theta_h) = \frac{\theta_m(\theta_h - \theta_2)}{\theta_2(\theta_h - \theta_m)}$

Equation

$$\begin{aligned} CR(\mathbf{p}_1, \mathbf{p}_{m1}, \mathbf{p}_2, \mathbf{p}_h) &= CR(0, a, 2a, \infty) \\ \frac{\theta_m(\theta_h - \theta_2)}{\theta_2(\theta_h - \theta_m)} &= 1/2 \end{aligned}$$

In the world

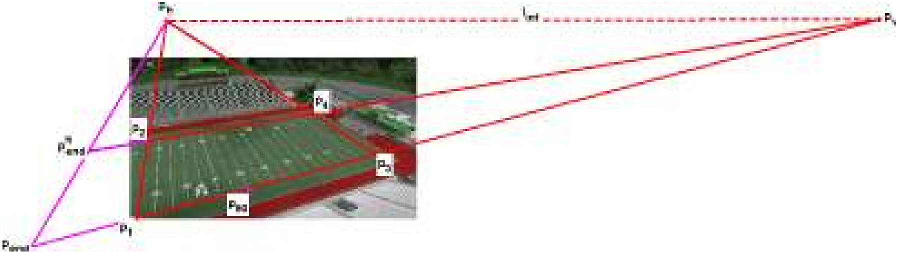
- $CR(0, a, 2a, \infty) = \frac{a\ \infty}{2a\ \infty} = \frac{1}{2}$
- a is the (unknow) half-length

Solution

- $\theta_m = \frac{\theta_2 \theta_h}{2\theta_h - \theta_2}$
- $\mathbf{p}_{m1} = \mathbf{p}_1 + \theta_m \, \bar{\mathbf{d}}_{12}$
- do the same for $\mathbf{p}_{m2}$

Cross Ratio Example - 6

Identify field bottom line



In the image

- Get the p_{50} point (field middle)
- $CR(\mathbf{p}_v, \mathbf{p}_3, \mathbf{p}_{50}, \mathbf{p}_{end}) = CR(0, \theta_3, \theta_m, \theta_{end})$

$$= \frac{\theta_3(\theta_{end} - \theta_m)}{\theta_m(\theta_{end} - \theta_3)}$$

In the world

- $CR(-\infty, 0, a, 2a) = \frac{\infty}{\infty}\frac{a}{2a} = \frac{1}{2}$
- a is the (unknow) half-length

Equation

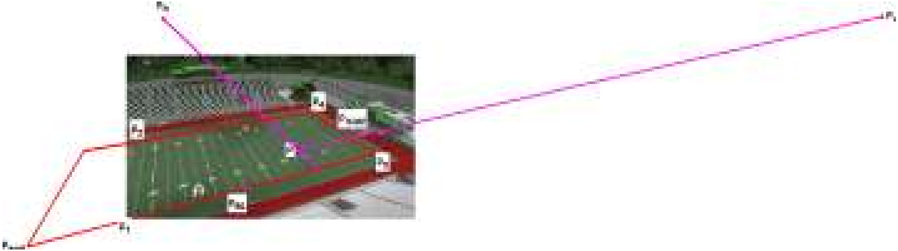
$$\begin{aligned} CR(\mathbf{p}_v, \mathbf{p}_3, \mathbf{p}_{50}, \mathbf{p}_{end}) &= CR(-\infty, 0, a, 2a) \\ \frac{\theta_3(\theta_{end} - \theta_m)}{\theta_m(\theta_{end} - \theta_3)} &= 1/2 \end{aligned}$$

Solution

- $\theta_{end} = \frac{\theta_m \theta_3}{2\theta_3 - \theta_m}$
- $\mathbf{p}_{end} = \mathbf{p}_v + \theta_{end}\overline{\mathbf{d}}_{v3}$
- $\mathbf{l}_{end} = \mathbf{p}_{end} \times \mathbf{p}_h$

Cross Ratio Example - 7

Calculate relative player P position



Origin

- in $\mathbf{p}_3$
- x towards $\mathbf{p}_4$
- y towards $\mathbf{p}_1$

Calculate

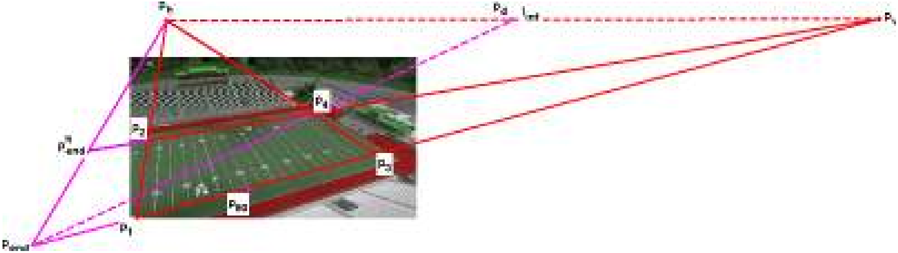
- $\mathbf{P}_x = (\mathbf{P} \times \mathbf{p}_v) \times (\mathbf{p}_3 \times \mathbf{p}_4)$
- $\mathbf{P}_y = (\mathbf{P} \times \mathbf{p}_h) \times (\mathbf{p}_1 \times \mathbf{p}_3)$

Cross Ratio

- $CR(\mathbf{p}_3, \mathbf{P}_x, \mathbf{p}_{mid2}, \mathbf{p}_4) = CR(0, x, \frac{1}{2}, 1)$
  - $\frac{\theta_x(\theta_4 - \theta_{mid2})}{\theta_{mid2}(\theta_4 - \theta_x)} = \frac{x}{1-x}$
- $CR(\mathbf{p}_3, \mathbf{P}_y, \mathbf{p}_{50}, \mathbf{p}_{end}) = CR(0, x, \frac{1}{2}, 1)$
  - $\frac{\theta_y(\theta_{end} - \theta_{50})}{\theta_{50}(\theta_{end} - \theta_y)} = \frac{x}{1-x}$

Cross Ratio Example - 8

Calculate vanishing of the diagonal



Calculate

- $\mathbf{l}_d = \mathbf{p}_{end} \times \mathbf{p}_4$
- $\mathbf{p}_d = \mathbf{l}_d \times \mathbf{l}'_\infty$

Last step - Affine reconstruction

Affine transformation

- $\mathbf{l}_\infty = [0,\, 0,\, 1]^T$ invariant
  but not point-wise!
- Consider $\mathbf{l}'_\infty = [l'_x,\, l'_y,\, l'_z]^T$
  image of $\mathbf{l}_\infty$
- Consider $\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l'_x & l'_y & l'_z \end{bmatrix}$
- Could be verified that $\mathbf{l}_\infty = \mathbf{H}^{-T}\mathbf{l}'_\infty$
- i.e., $\mathbf{p}_{aff} = \mathbf{H}\,\mathbf{p}_{img}$,
  $\mathbf{H}$ map points of the image to a affine transformation of the world

Source image

Affine reconstruction

Outline

Projective Geometry - 3D

Points

- Points $\mathbf{p}_e = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \in \mathbb{R}^3$
  in Cartesian coordinates
- $\mathbf{p}_h = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in \mathbb{R}^4$
  in homogeneous coordinates
- $\begin{cases} X & = & x/w \\ Y & = & y/w \\ Z & = & z/w \\ w & \neq & 0 \end{cases}$
- i.e., there is an arbitrary *scale factor*

Planes

- Planes $\pi = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4$
- $\mathbf{n} = \frac{\begin{bmatrix} a, & b, & c \end{bmatrix}^T}{\left\| \begin{bmatrix} a, & b, & c \end{bmatrix}^T \right\|}$
  unitary normal to the plane
- $\mathbf{p}_h \in \pi \iff \mathbf{p}_h^T \pi = \pi^T \mathbf{p}_h = 0$
- $\pi_\infty = \begin{bmatrix} 0, & 0, & 0, & 1 \end{bmatrix}^T$: plane at infinity
  contains all improper points

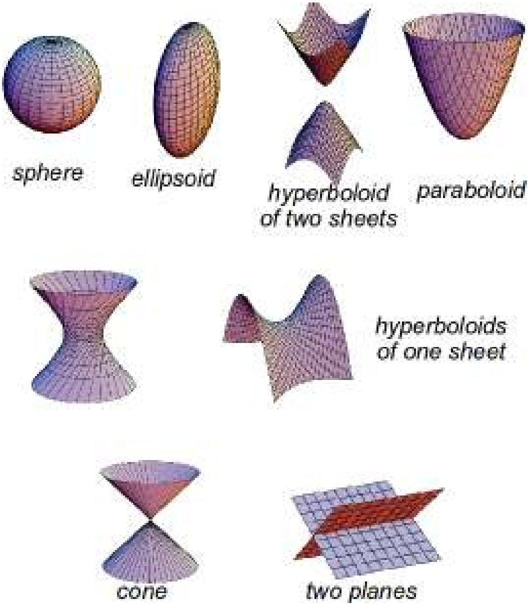
Quadrics

DEFINITION

- Quadratic polynomial equation
- *Quadric surface*
- Matrix form equation
  - $\mathbf{x}^T \mathbf{Q} \mathbf{x} = 0$
- $\mathbf{Q}$ is 4×4 symmetric

$\rightarrow$ $\mathbf{Q}$ is homogeneous too, i.e., 10 parameters, 9 D.O.F.

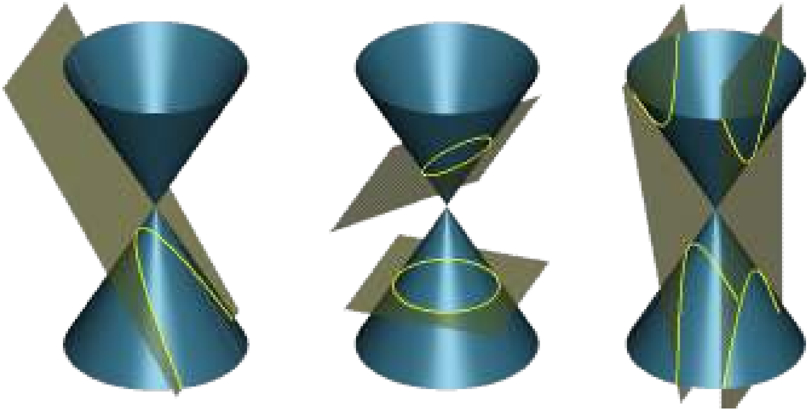
Quadrics - Summmary



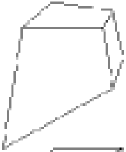
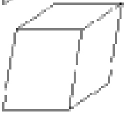
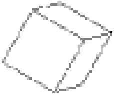
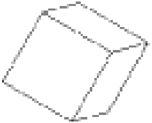
Quadrics & conics

INTERSECTION

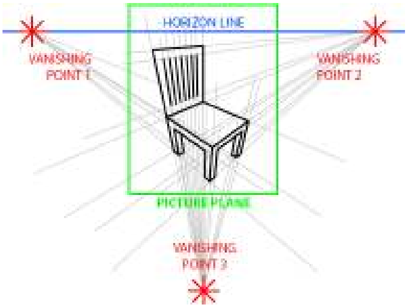
- $\mathbf{Q} \cap \pi \rightarrow$ conic
- Conics are planar sections of quadrics



Hierarchy of transformations

Projective 15dof	$\begin{bmatrix} \mathrm{A} & \mathrm{t} \\ \mathrm{v}^{\mathrm{T}} & v \end{bmatrix}$	
Affine 12dof	$\begin{bmatrix} \mathrm{A} & \mathrm{t} \\ 0^{\mathrm{T}} & 1 \end{bmatrix}$	
Similarity 7dof	$\begin{bmatrix} s\,\mathrm{R} & \mathrm{t} \\ 0^{\mathrm{T}} & 1 \end{bmatrix}$	
Euclidean 6dof	$\begin{bmatrix} \mathrm{R} & \mathrm{t} \\ 0^{\mathrm{T}} & 1 \end{bmatrix}$	

Vanishing points



VANISHING POINTS

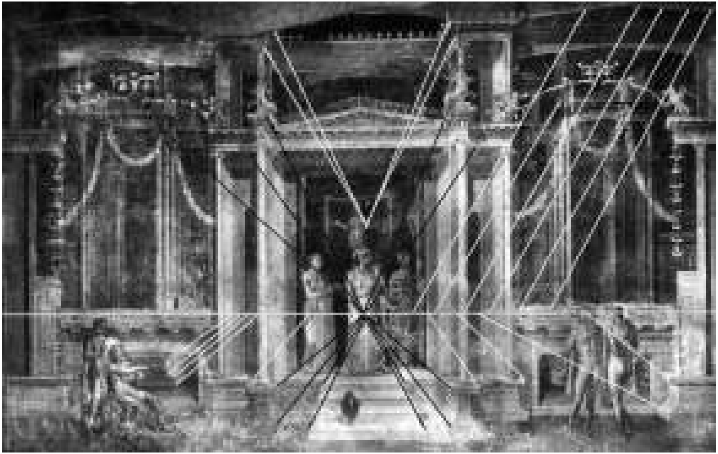
- π_∞ contains all the *directions*
- All the lines with the same direction intersect on π_∞ at the same point
- The vanishing point is the *image* of this intersection

VANISHING LINES

- Parallel planes intersect π_∞ in a common line
- The vanishing line is the *image* of this intersection
- e.g., the *horizon line* is the *image* of the intersection of the set of horizontal planes $\{\pi_H\}$ with π_∞

Art & Perspective - 1

Fresco in Pompeii - I B.C.



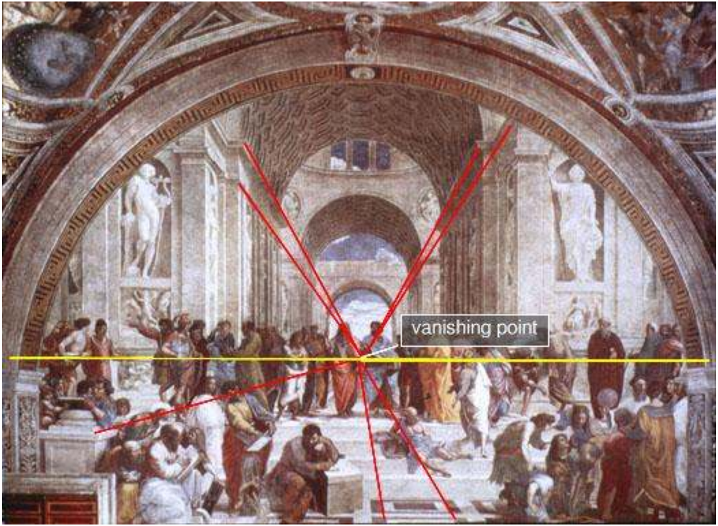
Partially correct perspective

The skill was lost during the middle ages,

it did not reappear in paintings until the Renaissance

Art & Perspective - 2

THE SCHOOL OF ATHENS - RAFFAELLO SANZIO - $\sim$ 1510



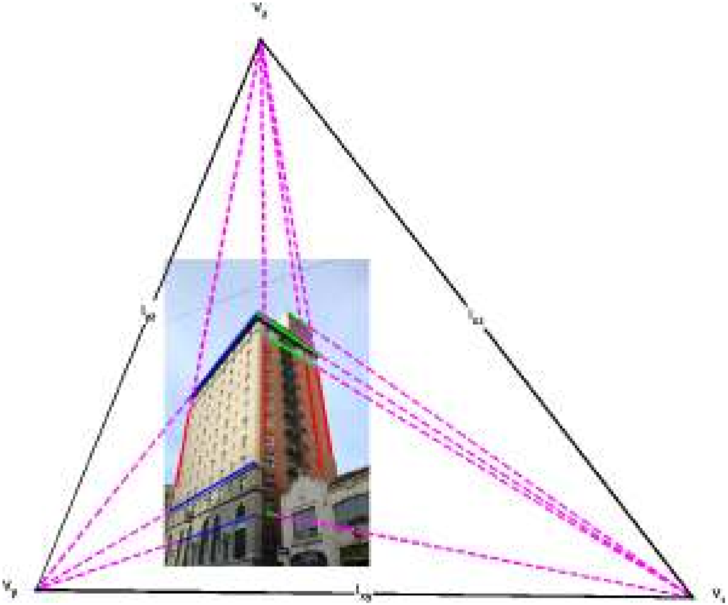
Correct perspective

Vanishing points example - 1

QUESTION

- Find the three vanishing point in the image
- Compute the horizon line in the image
- Compute others vanishing lines $\cdots$

Vanishing points example - 2



Outline

Reconstruction example - 1

Flagellazione di Cristo - Piero della Francesca - ∼ 1450

Reconstruction example - 2

TRINITY - MASACCIO - $\sim$ 1426

Reconstruction example - 3

A simple photo

Nice stuff with Projective geometry - 1

Felice Varini - *http://www.varini.org/*

Nice stuff with Projective geometry - 2

Felice Varini - *http://www.varini.org/*

Nice stuff with Projective geometry - 3

Felice Varini - *http://www.varini.org/*

Nice stuff with Projective geometry - 4

Felice Varini - *http://www.varini.org/*

Nice stuff with Projective geometry - 5

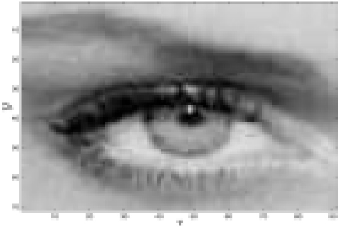
Felice Varini - *http://www.varini.org/*

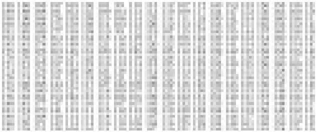
Outline

What is an image

Image

- Two-dimensional brightness array: $\mathbf{I}$
- $3\times$ two-dimensional array: $\mathbf{I}_R, \mathbf{I}_G, \mathbf{I}_B$
  - RGB: Red, Green, Blue
  - others: YUV, HSV, HSL, $\cdots$
- Ideal: $\mathbf{I} : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}_+$
- Discrete: $\mathbf{I} : \Omega \subset \mathbb{N}^2 \rightarrow \mathbb{R}_+^*$
  - e.g., $\Omega = [0, 639] \times [0, 479] \subset \mathbb{N}^2$
  - e.g., $\Omega = [1, 1024] \times [1, 768] \subset \mathbb{N}^2$
  - e.g., $\mathbb{R}_+^* = [0, 255] \subset \mathbb{N}$
  - e.g., $\mathbb{R}_+^* = [0, 1] \subset \mathbb{R}$
- $\mathbf{I}(x, y)$ is the intensity
- $\mathbf{I}$ result of 3D $\rightarrow$ 2D projection: *flat*





Camera

Optical system

- Set of lenses to direct light

  change in the direction of propagation

- CCD sensor

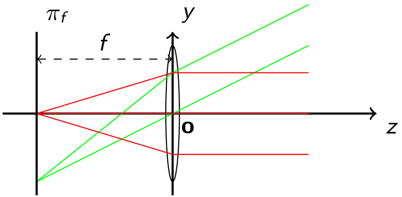
  integrate energy both

  - in time (exposure time)
  - in space (pixel size)

Thin lenses model

Thin lenses

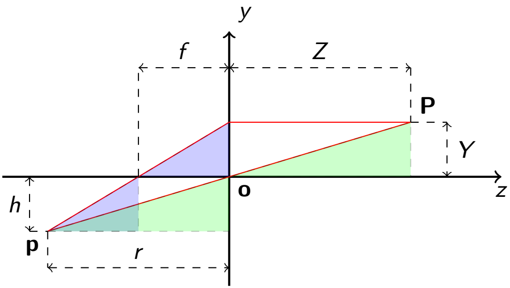
- Mathematical model
  - Optical axis (z)
  - Focal plane π_f ($\perp z$)
  - Optical center **o**
- Parameters
  - f distance **o**, π_f
- Property
  - Parallel rays converge π_f
  - Rays through **o** undeflected



Rays from scene

Image from a scene point **P**

- $\mathbf{P} = (Z, Y)$
- Ray through **o** undeflected
- Ray parallel to z cross in $(-f, 0)$



Similarities

- Blue triangles: $\dfrac{h}{Y} = \dfrac{r-f}{f}$
- Green triangles: $\dfrac{h}{Y} = \dfrac{r}{Z}$

Fresnel Law

- $\dfrac{1}{Z} + \dfrac{1}{r} = \dfrac{1}{f}$
- Note: $Z \to \infty \Rightarrow r \to f$

The image plane

Image plane $\pi_{\mathbf{I}}$

- Plane $\perp$ z at distance d

Blur Circle

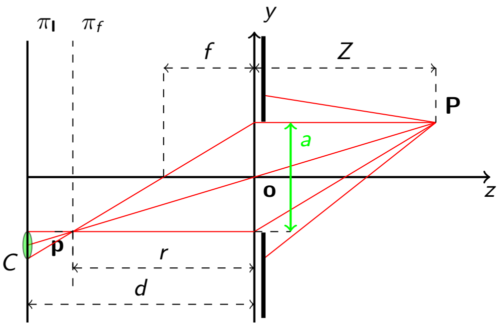
- If $d \neq r$

  image of **P** is a circle C

- Diameter of C:

$$\phi(C) = \frac{a(d-r)}{r}$$

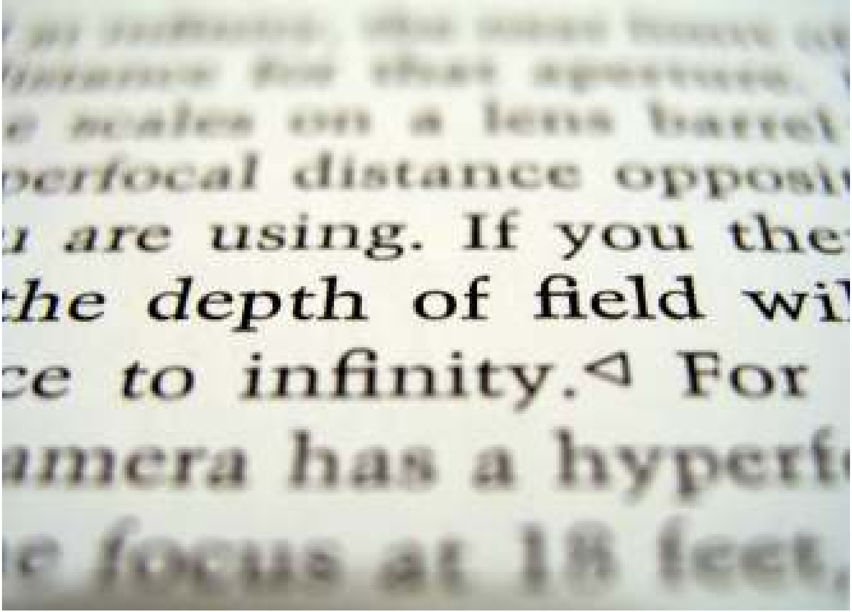
a is the *aperture*



Focused image

- $\phi(C) <$ pixel size
- Depth of field : range $[Z_1, Z_2]$: $\phi(C) <$ pixel size

Depth of field - Example 1



Depth of field - Example 2

THE SAME SCENE - DIFFERENT APERTURE

Outline

Pin hole model - Definition

Hypothesis

- $Z \gg a$
- $Z \gg f \rightarrow r \sim f$

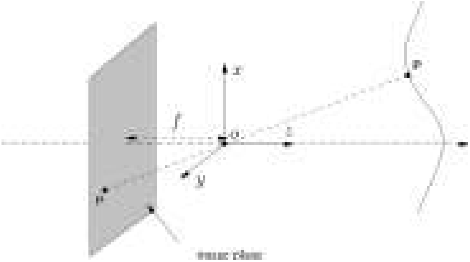
Image of $\mathbf{P}$

- $\mathbf{l}_{Po}$: line that join $\mathbf{P}$ and $\mathbf{o}$
- $\mathbf{p} = \pi_{\mathbf{I}} \cap \mathbf{l}_{Po}$

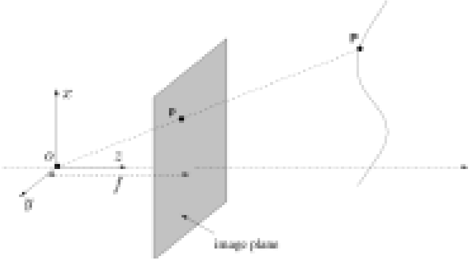
Notes

- $\mathbf{p}$ is the image of $\forall \mathbf{P}_i \in \mathbf{l}_{Po}$
- $\mathbf{l}_{Po}$: interpretation line of $\mathbf{p}$

Pin-Hole Model



Frontal Pin-Hole Model



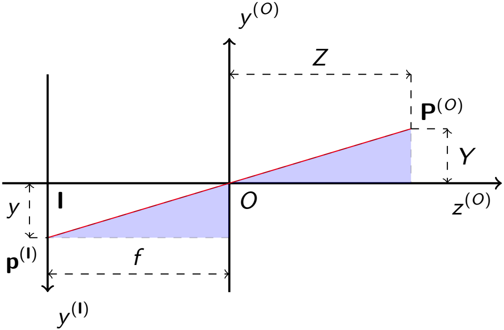
Pin hole model - Geometry

Given

- $\mathbf{P}^{(O)} = \left[X, Y, Z, 1\right]^T$
- $\mathbf{p}^{(\mathbf{I})} = \left[x, y, 1\right]^T$

Projection

- $y = f\dfrac{Y}{Z}$
- $x = f\dfrac{X}{Z}$
  look at the triangles



Note

- $\lambda\mathbf{P}^{(O)}$ projects on $\mathbf{p}^{(\mathbf{I})}$
- $\left[sX, sY, sZ, 1\right]^T$ projects on $\mathbf{p}^{(\mathbf{I})}$, $\forall s \neq 0$

Pin hole model - Matrix

Projection equations

- $y = f\dfrac{Y}{Z}$
- $x = f\dfrac{X}{Z}$

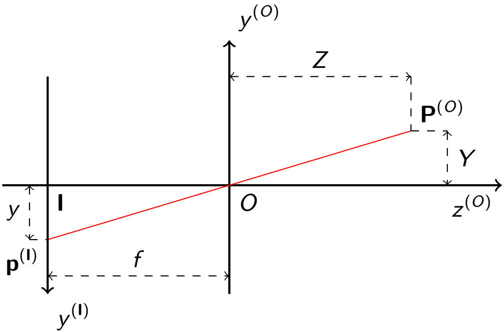
In matrix form

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix}$$

$$\mathbf{p}^{(\mathbf{I})} = \boldsymbol{\pi}\ \mathbf{P}^{(O)}$$

Define

- $\mathbf{K} = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix}$: intrinsic parameters
- $\pi = \begin{bmatrix} \mathbf{K} & \mathbf{0} \end{bmatrix}$: projection matrix



Pin hole model - Image coordinates - 1

Reference system on image

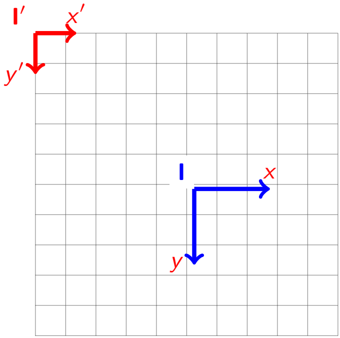
- $\mathbf{I}$: origin centered on $z^{(O)} \cap \pi_{\mathbf{I}}$
- $\mathbf{I}'$: origin centered top-left image
- $\mathbf{c}^{(\mathbf{I}')} = \left[\mathbf{c}_x, \mathbf{c}_y\right]^T$: position of $\mathbf{I}$ in $\mathbf{I}'$

Metric

- $\mathbf{I}$ metric
- $\mathbf{I}'$ in pixel
- $\mathbf{c}^{(\mathbf{I}')}$ in pixel

Definition

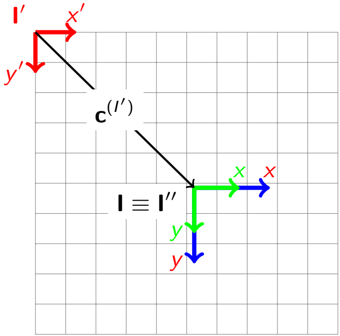
- $\left[0, 0\right]^{T(\mathbf{I})} \equiv \left[\mathbf{c}_x, \mathbf{c}_y\right]^{T(\mathbf{I}')}$: *principal point*
- Image of the optical center ($\mathbf{o}$) *or* $z^{(O)} \cap \pi_{\mathbf{I}}$



Pin hole model - Image coordinates - 2

Meters to pixels

- Consider $\mathbf{I}''$: origin on $\mathbf{I}$, *in pixel*
- Scale meters to pixels
  - $\mathbf{p}_x^{(\mathbf{I}'')} = \mathbf{s}_x \mathbf{p}_x^{(\mathbf{I})}$
  - $\mathbf{p}_y^{(\mathbf{I}'')} = \mathbf{s}_y \mathbf{p}_y^{(\mathbf{I})}$
- $\mathbf{s}_x = \frac{1}{d_x}$, d_x: width of a pixel $[m]$
- $\mathbf{s}_y = \frac{1}{d_y}$, d_y: height of a pixel $[m]$
- $\mathbf{s}_x = \mathbf{s}_y$: square pixel
- $\mathbf{p}^{(\mathbf{I}'')} = \begin{bmatrix} \mathbf{s}_x & 0 & 0 \\ 0 & \mathbf{s}_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{p}^{(\mathbf{I})}$



Translation

- $\mathbf{p}^{(\mathbf{I}')} = \begin{bmatrix} 1 & 0 & \mathbf{c}_x \\ 0 & 1 & \mathbf{c}_y \\ 0 & 0 & 1 \end{bmatrix} \mathbf{p}^{(\mathbf{I}'')}$

Pin hole model - Intrinsic camera matrix

Consider

- $\mathbf{p}^{(\mathbf{I})} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{P}^{(O)}$
- $\mathbf{p}^{(\mathbf{I}'')} = \begin{bmatrix} \mathbf{s}_x & 0 & 0 \\ 0 & \mathbf{s}_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{p}^{(\mathbf{I})}$
- $\mathbf{p}^{(\mathbf{I}')} = \begin{bmatrix} 1 & 0 & \mathbf{c}_x \\ 0 & 1 & \mathbf{c}_y \\ 0 & 0 & 1 \end{bmatrix} \mathbf{p}^{(\mathbf{I}'')}$

In one step

- $\mathbf{p}^{(\mathbf{I}')} = \begin{bmatrix} \mathbf{s}_x f & 0 & \mathbf{c}_x & 0 \\ 0 & \mathbf{s}_y f & \mathbf{c}_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{P}^{(O)}$

The intrinsic camera matrix

or *calibration matrix*

$$\mathbf{K} = \begin{bmatrix} f_x & s & \mathbf{c}_x \\ 0 & f_y & \mathbf{c}_y \\ 0 & 0 & 1 \end{bmatrix}$$

- f_x, f_y: focal lenght (in pixels)
  $f_x/f_y = s_x/s_y = a$: aspect ratio
- s: skew factor
  pixel not orthogonal
  usually 0 in modern cameras
- $\mathbf{c}_x$, $\mathbf{c}_y$: principal point (in pixel)
  usually $\neq$ half image size due to misalignment of CCD

Outline

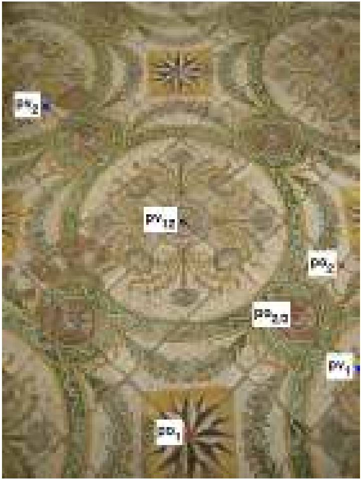
Exercise 1 - Tiles

Image source

Questions

- Identify the vanishing points
- using cross ratio
- i.e., without use parallel lines

Exercise 1 - Tiles - Solution



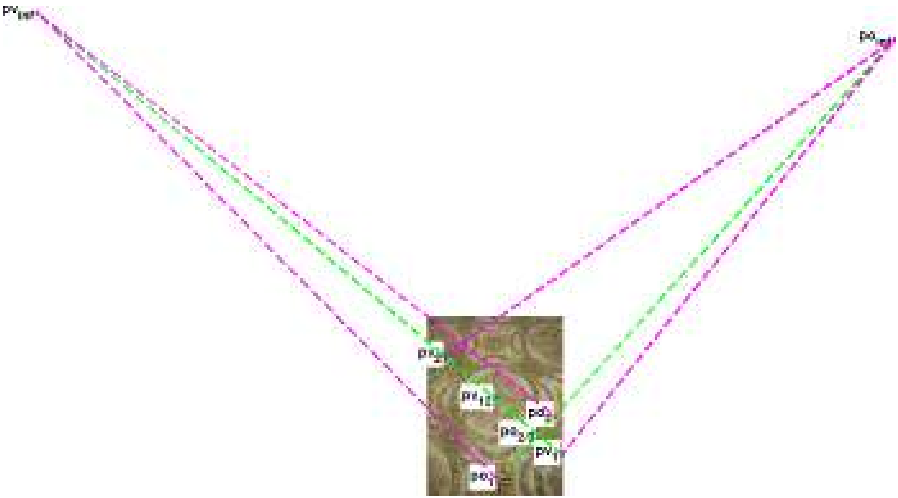
HORIZONTAL

- $CR(\mathbf{p}_{o1}, \mathbf{p}_{o23}, \mathbf{p}_{o2}, \mathbf{p}_{o\infty}) = CR(0, a2/3, a, \infty)$
- $CR(0, \theta_{23}, \theta_3, \theta_o) = 2/3$
- $\theta_o = \frac{-\theta_{23}\theta_3}{2\theta_3 - 3\theta_{23}}$
- $\mathbf{p}_{o\infty} = \mathbf{p}_{o1} + \theta_o \bar{\mathbf{d}}_o$

VERTICAL

- $CR(\mathbf{p}_{v1}, \mathbf{p}_{v12}, \mathbf{p}_{v2}, \mathbf{p}_{v\infty}) = CR(0, a, 2a, \infty)$
- $CR(0, \theta_{12}, \theta_{12}, \theta_v) = 1/2$
- $\theta_v = \frac{\theta_{12}(\theta_v - \theta_2)}{\theta_2(\theta_v - \theta_{12})}$
- $\mathbf{p}_{v\infty} = \mathbf{p}_{v1} + \theta_v \bar{\mathbf{d}}_v$

Exercise 1 - Tiles - Check



Magenta lines only for check correctness

Exercise 2 - Soccer field

Image source

Find

- Center of the goal-line
- Vanishing point of the goal-line

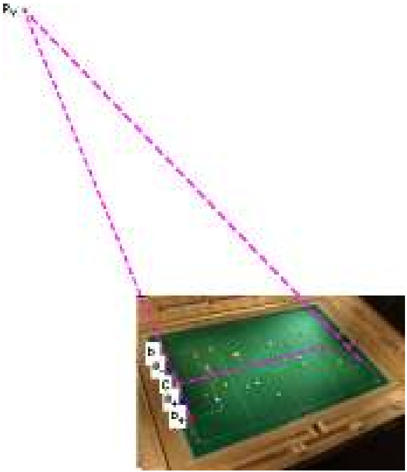
Solution

- 4 symmetric points
- a_-, a_+ and b_-, b_+

$$\begin{cases} CR(0, -a, a, \infty) & = & CR(\theta_c, \theta_{a_-}, \theta_{a_+}, \theta_v) \\ CR(0, -b, b, \infty) & = & CR(\theta_c, \theta_{b_-}, \theta_{b_+}, \theta_v) \end{cases}$$

- 2 equations, 2 unknown
- 4 solutions, only 2 are are valid

Exercise 2 - Soccer field



Magenta lines only for check correctness