Robotics - Localization - E.K.F.

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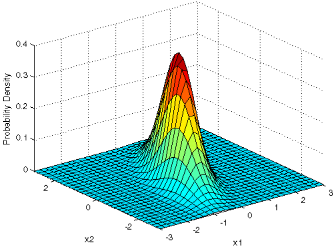
Outline

Outline

Gaussian Distribution Reminder

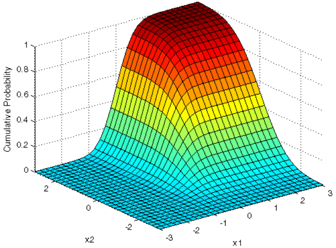
P.D.F. - $n = 2$



Multivariate Gaussian Distribution

- $\mathbf{x}$ is a vector
- $\mathcal{N}(\boldsymbol{\mu}, \Sigma)$, with
  - $\boldsymbol{\mu}$: $n \times 1$, mean vector
  - Σ: $n \times n$, covariance matrix
- Linear transformation:
  - $X \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$
  - $Y = \mathbf{A}X + \mathbf{b} \sim \mathcal{N}(\mathbf{A}\mu + \mathbf{b}, \mathbf{A}\Sigma\mathbf{A}^T)$

C.D.F. - $n = 2$



Kalman Filter - Introduction

Kalman Filter

- Was introduced by R. E. Kálmán in 1960
- It is a technique for *optimal* *filtering* and *prediction* in *Linear Gaussian System*
- Implements belief computation
  - for continuous state
  - distributed as a multivariate Gaussian
  - i.e., belief is represented by $\boldsymbol{\mu}$ and Σ
- The best studied technique for implementing Bayes Filters

Kalman Filter - Hypothesis

Kalman Filter and Gaussian beliefs

- Hypothesis that guarantee that *posterior belief* is Gaussian
  1. *State transition probability* is a linear function:
     $\mathbf{x}_t = \mathbf{A}_t \mathbf{x}_{t-1} + \mathbf{B}_t \mathbf{u}_t + \epsilon_t$, where
     - $\mathbf{x}_t$ is the state vector $[n \times 1]$
     - $\mathbf{u}_t$ is the control input $[m \times 1]$
     - $\epsilon_t \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_t)$ is a n-dimensional Gaussian random variable that models the uncertainty
     - $\mathbf{A}_t$ is the state transition matrix $[n \times n]$
     - $\mathbf{B}_t$ is the input transition matrix $[n \times m]$
  2. *Measurement probability* must be linear:
     $\mathbf{z}_t = \mathbf{C_t} \mathbf{x}_t + \delta_t$
     - $\mathbf{z}_t$ is the measurement vector $[k \times 1]$
     - $\mathbf{C}_t$ express the relation between state and measure $[k \times n]$
     - $\delta_t \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_t)$ is a k-dimensional Gaussian random variable that models the uncertainty
  3. *Initial belief* need to be normally distributed:
     $bel(x_0) \sim \mathcal{N}(\boldsymbol{\mu}_0, \Sigma_0)$

Kalman Filter - Algorithm

Algorithm Kalman_filter($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$):

$$
\text{①} \begin{cases} \bar{\mu}_t = A_t\ \mu_{t-1} + B_t\ u_t \\ \bar{\Sigma}_t = A_t\ \Sigma_{t-1}\ A_t^T + R_t \end{cases}
$$

$$
K_t = \bar{\Sigma}_t\ C_t^T (C_t\ \bar{\Sigma}_t\ C_t^T + Q_t)^{-1}
$$

$$
\text{②} \begin{cases} \mu_t = \bar{\mu}_t + K_t (z_t - C_t\ \bar{\mu}_t) \\ \Sigma_t = (I - K_t\ C_t)\ \bar{\Sigma}_t \end{cases}
$$

return μ_t, Σ_t

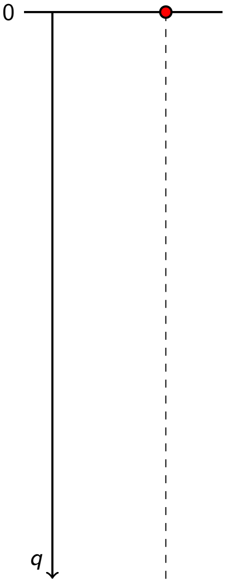
Two Step Algorithm

1. First Step - Prediction
   - Calculate the $\overline{bel}(x_t)$: *mean* ($\overline{\mu}$) and covariance ($\overline{\Sigma}$)
   - Application of Gaussian properties

2. Second Step - Update
   - Calculate the $bel(x_t)$: *mean* ($\boldsymbol{\mu}$) and covariance (Σ)
   - Using the *Kalman Gain* ($\mathbf{K}_t$) on the *innovation*, i.e. difference of
     - Measure: z_t
     - Expected Measure: $\mathbf{C}_t \mu_t$

Outline

Kalman Filter - Example - Problem



Falling body

- Start at 0m with 0 m/s speed
- Standard deviation on position is 0[m]
- Standard deviation on speed is 0[m/s]
- A sensor measure the altitude in millimeters with a stochastic error of 100mm
- During the fall, stochastic errors affect
  - altitude, with std. dev. of 0.01m
  - speed, with std. dev. of 0.005m/s

Kalman Filter - Example - Definitions

Definitions

- State: $x_t = \left[q_t, v_t\right]^T$, altitude and speed
- Input: $u_t = g$, gravity
- Constants
  - $b = 0.0025$, friction coefficient
  - $\Delta t = 0.001$s, period of discrete time step

Initial State

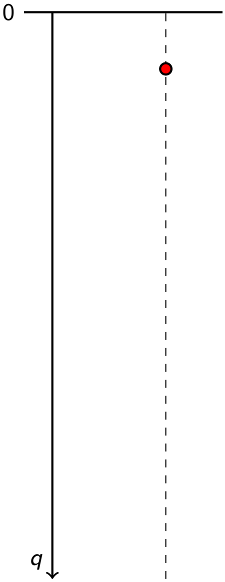
- $\boldsymbol{\mu}_0 = \left[0, 0\right]^T, \; \Sigma_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

State Transition

- $q_t = q_{t-1} + v_{t-1}\Delta t + \epsilon_{1t}$
- $v_t = v_{t-1} + g\Delta t - bv_{t-1} + \epsilon_{2t}$
- Matrix form:

$$\underbrace{\begin{bmatrix} q_t \\ v_t \end{bmatrix}}_{\mathbf{x}_t} = \underbrace{\begin{bmatrix} 1 & \Delta t \\ 0 & 1-b \end{bmatrix}}_{\mathbf{A}_t} \underbrace{\begin{bmatrix} q_{t-1} \\ v_{t-1} \end{bmatrix}}_{\mathbf{x}_{t-1}} + \underbrace{\begin{bmatrix} 0 \\ \Delta t \end{bmatrix}}_{\mathbf{B}_t} \underbrace{g}_{\mathbf{u}_t} + \underbrace{\begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{bmatrix}}_{\epsilon_t}$$

- $\mathbf{R}_t = \begin{bmatrix} 0.01^2 & 0 \\ 0 & 0.005^2 \end{bmatrix}$

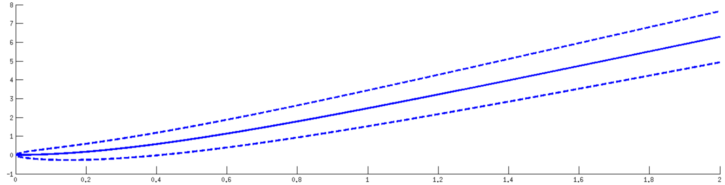


Kalman Filter - Example - Prediction

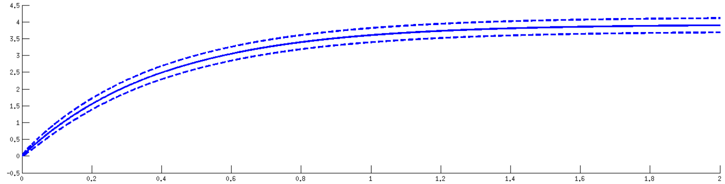
Only Predictions Effects

- $\overline{\boldsymbol{\mu}}_t = \mathbf{A}\boldsymbol{\mu}_{t-1} + \mathbf{B}\mathbf{u}_1$
- $\overline{\Sigma}_t = \mathbf{A}\Sigma_{t-1}\mathbf{A}^T + \mathbf{R}$

Altitude prediction



Speed prediction

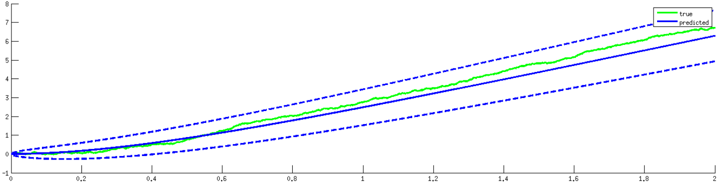


Dashed lines are $\overline{\boldsymbol{\mu}}_t \pm 3 * \sqrt{\text{diag}(\overline{\Sigma})_t}$

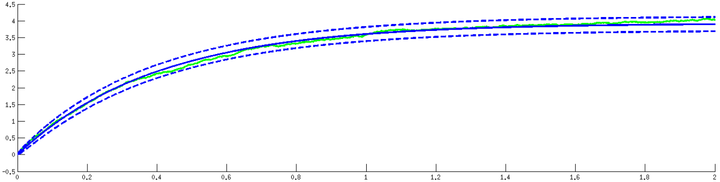
Kalman Filter - Example - Prediction vs True State

What's happen in the real world - State Transition

Altitude prediction vs true altitude



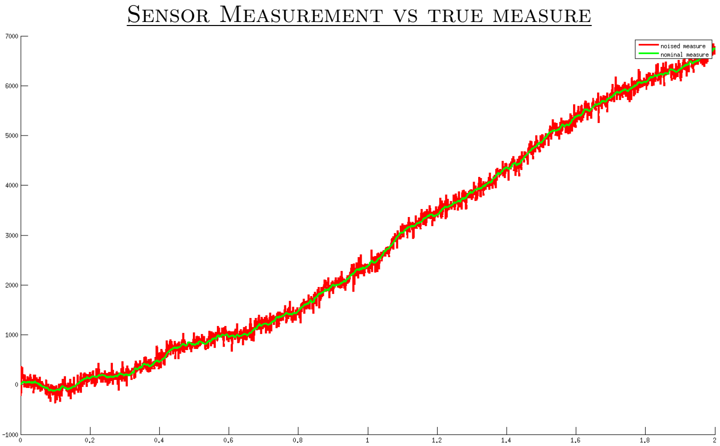
Speed prediction vs true speed



Real Values is contained in the 3σ confidence interval

Kalman Filter - Example - Sensor Measurement vs True Measure

What's happen in the real world - Measurement



Kalman Filter - Example - Update Step

Measurement Model

- $z_t = \mathbf{C}_t \mathbf{x}_t + \delta_t$
- $z_t = 1000 \cdot q_t + \delta_t$
- $\mathbf{C}_t = \begin{bmatrix} 1000 & 0 \end{bmatrix}$
- $\mathbf{Q}_t = \begin{bmatrix} 100^2 \end{bmatrix}$

The Update Step

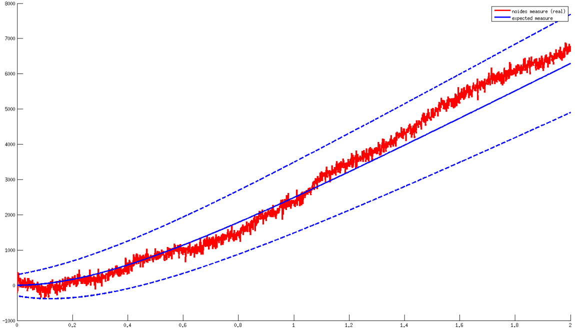
$$K_t = \bar{\Sigma}_t\, C_t^T (C_t\, \bar{\Sigma}_t\, C_t^T + Q_t)^{-1}$$
$$\mu_t = \bar{\mu}_t + K_t(z_t - C_t\, \bar{\mu}_t)$$
$$\Sigma_t = (I - K_t\, C_t)\, \bar{\Sigma}_t$$

An Intermediate Step

- Let's define
  - $\mathbf{y}_t = \mathbf{C}_t \overline{\boldsymbol{\mu}}_t$
  - $\mathbf{S}_t = \mathbf{C}_t \overline{\Sigma}_t \mathbf{C}_t^T + \mathbf{Q}_t$
- They represent a *measurement prediction*
- i.e., the (Gaussian) probability of the expected measurement $\sim \mathcal{N}(\mathbf{y}_t, \mathbf{S}_t)$

Kalman Filter - Example - Measurement Step

MEASUREMENT PREDICTION



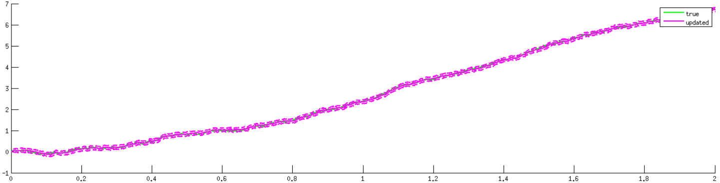
Dashed lines are $\mathbf{y}_t \pm 3 * \sqrt{\text{diag}(\mathbf{S})_t}$

Up to now, no Update Step!!

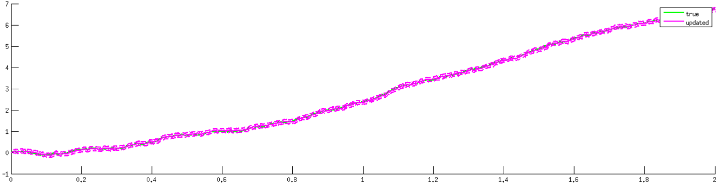
Kalman Filter - Example - Run the update

Prediction + Update

Altitude vs true altitude



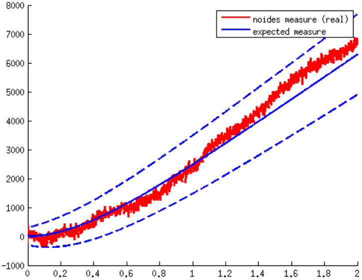
Speed vs true speed



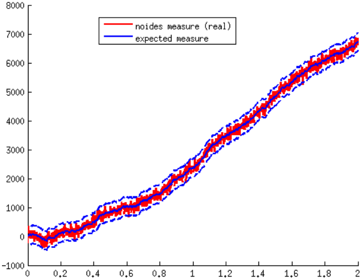
The filter state track the real value!

Kalman Filter - Example - Comparisons

Measurement Prediction No Update



Measurement Prediction With Update

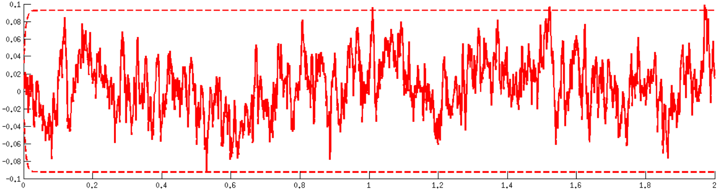


Kalman Filter - Example - Errors

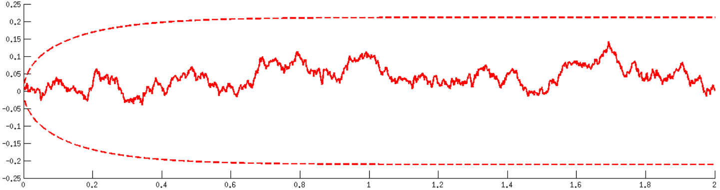
Error on State

- $\mathbf{x}_t^* - \mu_t$: difference between true value and estimated

Altitude Error



Speed Error



Outline

Extended Kalman Filter

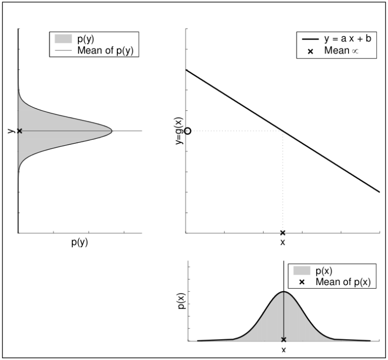
KALMAN FILTER MAIN ISSUE

- Works with linear systems
- Most real systems are modelled by nonlinear functions
- Can we “extend” it to treat non linear system?
- Yes, but under some hypothesis and with some drawbacks
- How to extend? $\rightarrow$ *local linearizaton*

Gaussian Variable Transformation - 1

<u>GIVEN</u> $x \sim \mathcal{N}(\mu, \sigma^2)$

$$y = ax + b$$

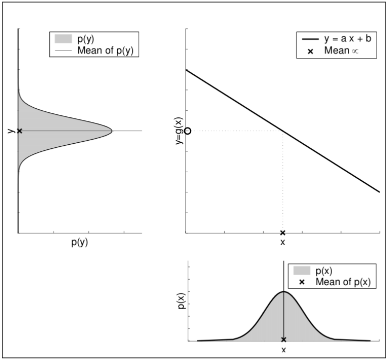


$$y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$$

Gaussian Variable Transformation - 1

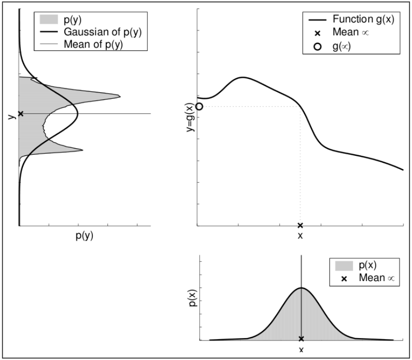
<u>GIVEN</u> $x \sim \mathcal{N}(\mu, \sigma^2)$

$y = ax + b$



$y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$

$y = g(x)$

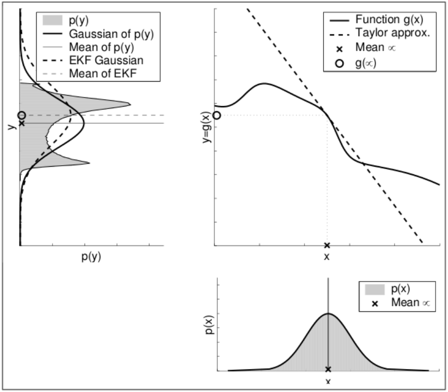


$y \nsim \mathcal{N}(\cdots)$

The maintenance of a good Gaussian approximation depends on the shape of $g(x)$

Gaussian Variable Transformation - 2

Linearization via Taylor Expansion of $g(x)$

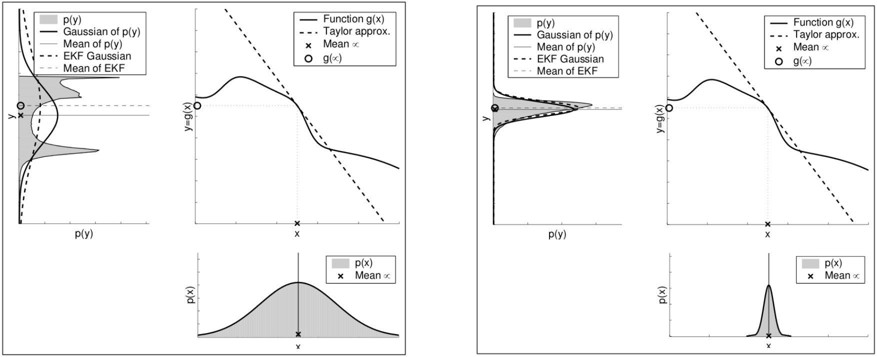


$$y \sim \mathcal{N}(\cdots)$$

The maintenance of a good Gaussian approximation depends on the linearization point

Gaussian Variable Transformation - 3

Linearization via Taylor Expansion of $g(x)$



The maintenance of a good Gaussian approximation depends on the variance of x

KF to EKF

KF

State transition

$$\mathbf{x}_t = \mathbf{A}_t \mathbf{x}_{t-1} + \mathbf{B}_t \mathbf{u}_t + \epsilon_t$$

Measurement probability

$$\mathbf{z}_t = \mathbf{C_t} \mathbf{x}_t + \delta_t$$

EKF

State transition

$$\mathbf{x}_t = g(\mathbf{x}_{t-1}, \mathbf{u}_t, \epsilon_t)$$

Measurement probability

$$\mathbf{z}_t = h(\mathbf{x}_t, \delta_t)$$

KF to EKF

KF

State transition

$$\mathbf{x}_t = \mathbf{A}_t \mathbf{x}_{t-1} + \mathbf{B}_t \mathbf{u}_t + \epsilon_t$$

Measurement probability

$$\mathbf{z}_t = \mathbf{C}_t \mathbf{x}_t + \delta_t$$

Prediction Step

$$\overline{\boldsymbol{\mu}}_t = \mathbf{A}_t \boldsymbol{\mu}_{t-1} + \mathbf{B}_t \mathbf{u}_t$$
$$\overline{\Sigma}_t = \mathbf{A}_t \Sigma_{t-1} \mathbf{A}_t^T + \mathbf{R}_t$$

Update Step

$$\mathbf{K}_t = \overline{\Sigma}_t \mathbf{C}_t^T (\mathbf{C}_t \overline{\Sigma}_t \mathbf{C}_t^T + \mathbf{Q}_t)^{-1}$$
$$\boldsymbol{\mu}_t = \overline{\boldsymbol{\mu}}_t - \mathbf{K}_t(\mathbf{z}_t + \mathbf{C}_t \overline{\boldsymbol{\mu}}_t)$$
$$\Sigma_t = (I - \mathbf{K}_t \mathbf{C}_t)\overline{\Sigma}_t$$

EKF

State transition

$$\mathbf{x}_t = g(\mathbf{x}_{t-1}, \mathbf{u}_t, \epsilon_t)$$

Measurement probability

$$\mathbf{z}_t = h(\mathbf{x}_t, \delta_t)$$

Prediction Step

$$\overline{\boldsymbol{\mu}}_t = g(\boldsymbol{\mu}_{t-1}, \mathbf{u}_t, 0)$$
$$\overline{\Sigma}_t = \mathbf{G}_t \Sigma_{t-1} \mathbf{G}_t^T + \mathbf{N}_t \mathbf{R}_t \mathbf{N}_t^T$$

Update Step

$$\mathbf{K}_t = \overline{\Sigma}_t \mathbf{H}_t^T (\mathbf{H}_t \overline{\Sigma}_t \mathbf{H}_t^T + \mathbf{M}_t \mathbf{Q}_t \mathbf{M}_t^T)^{-1}$$
$$\boldsymbol{\mu}_t = \overline{\boldsymbol{\mu}}_t + \mathbf{K}_t(\mathbf{z}_t - h(\overline{\boldsymbol{\mu}}_t, 0))$$
$$\Sigma_t = (I - \mathbf{K}_t \mathbf{H}_t)\overline{\Sigma}_t$$

EKF - Jacobians

EKF

State transition

$$\mathbf{x}_t = g(\mathbf{x}_{t-1}, \mathbf{u}_t, \epsilon_t)$$

Measurement probability

$$\mathbf{z}_t = h(\mathbf{x}_t, \delta_t)$$

Prediction Step

$$\overline{\boldsymbol{\mu}}_t = g(\boldsymbol{\mu}_{t-1}, \mathbf{u}_t, 0)$$
$$\overline{\Sigma}_t = \mathbf{G}_t \Sigma_{t-1} \mathbf{G}_t^T + \mathbf{N}_t \mathbf{R}_t \mathbf{N}_t^T$$

Update Step

$$\mathbf{K}_t = \overline{\Sigma}_t \mathbf{H}_t^T (\mathbf{H}_t \overline{\Sigma}_t \mathbf{H}_t^T + \mathbf{M}_t \mathbf{Q}_t \mathbf{M}_t^T)^{-1}$$
$$\boldsymbol{\mu}_t = \overline{\boldsymbol{\mu}}_t + \mathbf{K}_t(\mathbf{z}_t - h(\overline{\boldsymbol{\mu}}_t, 0))$$
$$\Sigma_t = (I - \mathbf{K}_t \mathbf{H}_t)\overline{\Sigma}_t$$

Jacobians

- $\mathbf{G}_t = \left.\frac{\partial g(\mathbf{x},\mathbf{u},\epsilon)}{\partial \mathbf{x}}\right|_{\mathbf{x}=\boldsymbol{\mu}_{t-1},\mathbf{u}=\mathbf{u}_t,\epsilon=0}$
  derivate of the state transition function w.r.t. state variables
- $\mathbf{N}_t = \left.\frac{\partial g(\mathbf{x},\mathbf{u},\epsilon)}{\partial \epsilon}\right|_{\mathbf{x}=\boldsymbol{\mu}_{t-1},\mathbf{u}=\mathbf{u}_t,\epsilon=0}$
  derivate of the state transition function w.r.t. noise variables
- $\mathbf{H}_t = \left.\frac{\partial h(\mathbf{x},\delta)}{\partial \mathbf{x}}\right|_{\mathbf{x}=\overline{\boldsymbol{\mu}}_{t-1},\delta=0}$
  derivate of the measurement function w.r.t. state variables
- $\mathbf{M}_t = \left.\frac{\partial h(\mathbf{x},\delta)}{\partial \delta}\right|_{\mathbf{x}=\overline{\boldsymbol{\mu}}_{t-1},\delta=0}$
  derivate of the measurement function w.r.t. noise variables

Outline

Localization with E.K.F.

Localization

- EKF can treat the *pose tracking* problem
- We can consider *uncertainty* (or *belief*) locally gaussian

State transition - Prediction

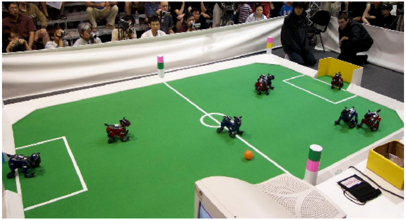
- The next robot position
  using motion information

Measurement - Update

- Sense of a *map landmark*
  e.g., sense distance and angle

Map and correspondences

- Position of landmarks in world coordinates
- Landmarks are uniquely identifiable
  e.g., different colors

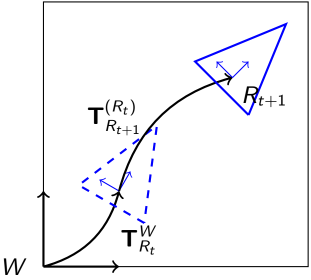


Prediction step - Robot motion - 1

State

- The robot position and orientation (2D)
  - $[x, y]$: the robot position
  - θ: orientation

in world reference frame: $\mathbf{T}^{W}_{R_t}$



State prediction

- Input $\mathbf{u}$: $[\Delta x, \Delta y, \Delta\theta] = [v_x \Delta t, v_y \Delta t, \omega \Delta t]$
- Relative motion: $\mathbf{T}^{R_t}_{R_{t+1}}(\mathbf{u})$
- Prediction: $\mathbf{T}^{W}_{R_{t+1}} = \mathbf{T}^{W}_{R_t} \cdot \mathbf{T}^{R_t}_{R_{t+1}}$

$$= \begin{bmatrix} \cos(\theta_t) & -\sin(\theta_t) & x_t \\ \sin(\theta_t) & \cos(\theta_t) & y_t \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos(\Delta\theta) & -\sin(\Delta\theta) & \Delta x \\ \sin(\Delta\theta) & \cos(\Delta\theta) & \Delta y \\ 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \cos(\theta_t + \Delta\theta) & -\sin(\theta_t + \Delta\theta) & \cos(\theta_t)\Delta x - \sin(\theta_t)\Delta y + x_t \\ \sin(\theta_t + \Delta\theta) & \cos(\theta_t + \Delta\theta) & \sin(\theta_t)\Delta x + \cos(\theta_t)\Delta y + y_t \\ 0 & 0 & 1 \end{bmatrix}$$

Prediction step - Robot motion - 2

State Prediction

- $\mathbf{x}_t = g(\mathbf{x}_{t-1}, \mathbf{u}_t, 0)$ (no noise)
- $$\begin{cases} x_{t+1} = \cos(\theta_t)\Delta x - \sin(\theta_t)\Delta y + x_t \\ y_{t+1} = \sin(\theta_t)\Delta x + \cos(\theta_t)\Delta y + y_t \\ \theta_{t+1} = \theta_t + \Delta\theta \end{cases}$$

Prediction step - Robot motion - 2

State Prediction

- $\mathbf{x}_t = g(\mathbf{x}_{t-1}, \mathbf{u}_t, 0)$ (no noise)
- $$\begin{cases} x_{t+1} = \cos(\theta_t)\Delta x - \sin(\theta_t)\Delta y + x_t \\ y_{t+1} = \sin(\theta_t)\Delta x + \cos(\theta_t)\Delta y + y_t \\ \theta_{t+1} = \theta_t + \Delta\theta \end{cases}$$

Noise Introduction

- Suppose that noise affects inputs
- $$\begin{cases} \tilde{\Delta x} = \Delta x + \epsilon_x \\ \tilde{\Delta y} = \Delta x + \epsilon_y \\ \tilde{\Delta \theta} = \Delta\theta + \epsilon_\theta \end{cases}$$
- $\boldsymbol{\epsilon} = [\epsilon_x, \epsilon_y, \epsilon_\theta] \sim \mathcal{N}(0, \Sigma_\epsilon)$
- $$\begin{cases} x_{t+1} = \cos(\theta_t)\tilde{\Delta x} - \sin(\theta_t)\tilde{\Delta y} + x_t \\ y_{t+1} = \sin(\theta_t)\tilde{\Delta x} + \cos(\theta_t)\tilde{\Delta y} + y_t \\ \theta_{t+1} = \tilde{\theta}_t + \Delta\theta \end{cases}$$

Prediction step - Robot motion - 2

State Prediction

- $\mathbf{x}_t = g(\mathbf{x}_{t-1}, \mathbf{u}_t, 0)$ (no noise)
- $\begin{cases} x_{t+1} = \cos(\theta_t)\Delta x - \sin(\theta_t)\Delta y + x_t \\ y_{t+1} = \sin(\theta_t)\Delta x + \cos(\theta_t)\Delta y + y_t \\ \theta_{t+1} = \theta_t + \Delta\theta \end{cases}$

Noise Introduction

- Suppose that noise affects inputs
- $\begin{cases} \tilde{\Delta x} = \Delta x + \epsilon_x \\ \tilde{\Delta y} = \Delta x + \epsilon_y \\ \tilde{\Delta\theta} = \Delta\theta + \epsilon_\theta \end{cases}$
- $\boldsymbol{\epsilon} = [\epsilon_x, \epsilon_y, \epsilon_\theta] \sim \mathcal{N}(0, \Sigma_\epsilon)$
- $\begin{cases} x_{t+1} = \cos(\theta_t)\tilde{\Delta x} - \sin(\theta_t)\tilde{\Delta y} + x_t \\ y_{t+1} = \sin(\theta_t)\tilde{\Delta x} + \cos(\theta_t)\tilde{\Delta y} + y_t \\ \theta_{t+1} = \tilde{\theta}_t + \Delta\theta \end{cases}$

Jacobians

- $\mathbf{G}_t = \left.\frac{\partial g(\mathbf{x},\mathbf{u},\epsilon)}{\partial \mathbf{x}}\right|_{\mathbf{x}=\boldsymbol{\mu}_{t-1},\mathbf{u}=\mathbf{u}_t,\epsilon=0}$

$$= \begin{bmatrix} \frac{\partial g_1(\mathbf{x},\mathbf{u},\epsilon)}{\partial x} & \frac{\partial g_1(\mathbf{x},\mathbf{u},\epsilon)}{\partial y} & \frac{\partial g_1(\mathbf{x},\mathbf{u},\epsilon)}{\partial \theta} \\ \frac{\partial g_2(\mathbf{x},\mathbf{u},\epsilon)}{\partial x} & \frac{\partial g_2(\mathbf{x},\mathbf{u},\epsilon)}{\partial y} & \frac{\partial g_2(\mathbf{x},\mathbf{u},\epsilon)}{\partial \theta} \\ \frac{\partial g_3(\mathbf{x},\mathbf{u},\epsilon)}{\partial x} & \frac{\partial g_3(\mathbf{x},\mathbf{u},\epsilon)}{\partial y} & \frac{\partial g_3(\mathbf{x},\mathbf{u},\epsilon)}{\partial \theta} \end{bmatrix}$$

- $\frac{\partial g_1(\mathbf{x},\mathbf{u},\epsilon)}{\partial \mathbf{x}} = 1, \quad \frac{\partial g_1(\mathbf{x},\mathbf{u},\epsilon)}{\partial \mathbf{y}} = 0$
- $\frac{\partial g_1(\mathbf{x},\mathbf{u},\epsilon)}{\partial \theta} = -\sin(\theta)\Delta x - \cos(\theta)\Delta y$

$$= \begin{bmatrix} 1 & 0 & -\sin(\theta)\Delta x - \cos(\theta)\Delta y \\ 0 & 1 & \cos(\theta)\Delta x - \sin(\theta)\Delta y \\ 0 & 0 & 1 \end{bmatrix}$$

Prediction step - Robot motion - 2

State Prediction

- $\mathbf{x}_t = g(\mathbf{x}_{t-1}, \mathbf{u}_t, 0)$ (no noise)
- $$\begin{cases} x_{t+1} = \cos(\theta_t)\Delta x - \sin(\theta_t)\Delta y + x_t \\ y_{t+1} = \sin(\theta_t)\Delta x + \cos(\theta_t)\Delta y + y_t \\ \theta_{t+1} = \theta_t + \Delta\theta \end{cases}$$

Noise Introduction

- Suppose that noise affects inputs
- $$\begin{cases} \tilde{\Delta x} = \Delta x + \epsilon_x \\ \tilde{\Delta y} = \Delta x + \epsilon_y \\ \tilde{\Delta\theta} = \Delta\theta + \epsilon_\theta \end{cases}$$
- $\boldsymbol{\epsilon} = [\epsilon_x, \epsilon_y, \epsilon_\theta] \sim \mathcal{N}(0, \Sigma_\epsilon)$
- $$\begin{cases} x_{t+1} = \cos(\theta_t)\tilde{\Delta x} - \sin(\theta_t)\tilde{\Delta y} + x_t \\ y_{t+1} = \sin(\theta_t)\tilde{\Delta x} + \cos(\theta_t)\tilde{\Delta y} + y_t \\ \theta_{t+1} = \tilde{\theta}_t + \Delta\theta \end{cases}$$

Jacobians

- $\mathbf{G}_t = \left.\frac{\partial g(\mathbf{x},\mathbf{u},\epsilon)}{\partial \mathbf{x}}\right|_{\mathbf{x}=\boldsymbol{\mu}_{t-1},\mathbf{u}=\mathbf{u}_t,\epsilon=0}$

$$= \begin{bmatrix} \frac{\partial g_1(\mathbf{x},\mathbf{u},\epsilon)}{\partial x} & \frac{\partial g_1(\mathbf{x},\mathbf{u},\epsilon)}{\partial y} & \frac{\partial g_1(\mathbf{x},\mathbf{u},\epsilon)}{\partial \theta} \\ \frac{\partial g_2(\mathbf{x},\mathbf{u},\epsilon)}{\partial x} & \frac{\partial g_2(\mathbf{x},\mathbf{u},\epsilon)}{\partial y} & \frac{\partial g_2(\mathbf{x},\mathbf{u},\epsilon)}{\partial \theta} \\ \frac{\partial g_3(\mathbf{x},\mathbf{u},\epsilon)}{\partial x} & \frac{\partial g_3(\mathbf{x},\mathbf{u},\epsilon)}{\partial y} & \frac{\partial g_3(\mathbf{x},\mathbf{u},\epsilon)}{\partial \theta} \end{bmatrix}$$

- $\frac{\partial g_1(\mathbf{x},\mathbf{u},\epsilon)}{\partial \mathbf{x}} = 1, \quad \frac{\partial g_1(\mathbf{x},\mathbf{u},\epsilon)}{\partial \mathbf{y}} = 0$
- $\frac{\partial g_1(\mathbf{x},\mathbf{u},\epsilon)}{\partial \theta} = -\sin(\theta)\Delta x - \cos(\theta)\Delta y$

$$= \begin{bmatrix} 1 & 0 & -\sin(\theta)\Delta x - \cos(\theta)\Delta y \\ 0 & 1 & \cos(\theta)\Delta x - \sin(\theta)\Delta y \\ 0 & 0 & 1 \end{bmatrix}$$

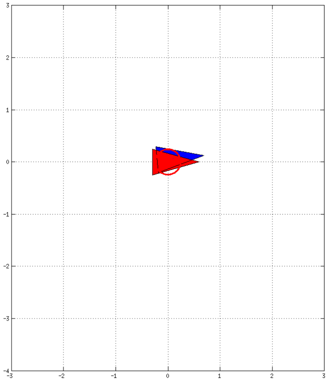
- $\mathbf{N}_t = \left.\frac{\partial g(\mathbf{x},\mathbf{u},\epsilon)}{\partial \epsilon}\right|_{\mathbf{x}=\boldsymbol{\mu}_{t-1},\mathbf{u}=\mathbf{u}_t,\epsilon=0}$

$$= \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Initial state

EKF STATE

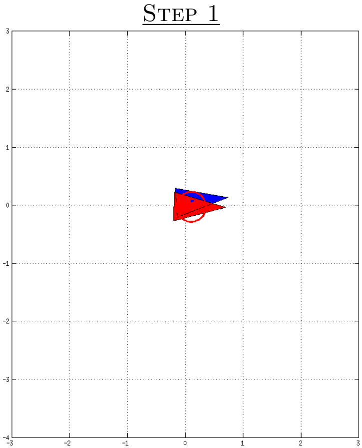
- $\mu_0 = [0, 0, 0]$
- $\Sigma_0 = \begin{bmatrix} 0.1^2 & 0 & 0 \\ 0 & 0.1^2 & 0 \\ 0 & 0 & \text{deg2rad}(10)^2 \end{bmatrix}$
- Robot is in the origin
- But we have some uncertainty on its position and orientation
- e.g., Robot real pose is $[0.094, 0.069, 5.2769°]$
- $\mathbf{C} = k \cdot \Sigma^{-1}_{0_{[1:2,1:2]}}$ confidence ellipse



blue: true position

red: estimated position (μ)

Only Prediction - 1



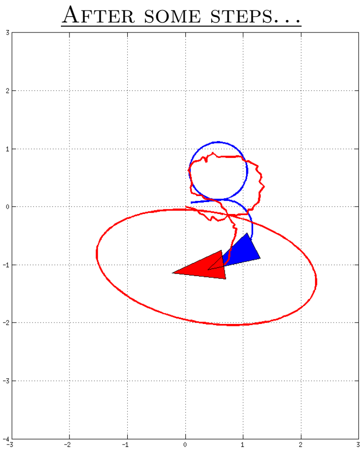
a (noisy) input arrives, the prediction step is performed

$$\overline{\boldsymbol{\mu}}_t = g(\boldsymbol{\mu}_{t-1}, \mathbf{u}_t, 0)$$
$$\overline{\Sigma}_t = \mathbf{H}_t \Sigma_{t-1} \mathbf{H}_t^T + \mathbf{N}_t \mathbf{R}_t \mathbf{N}_t^T$$

blue: true position

red: estimated position (μ)

Only Prediction - 2



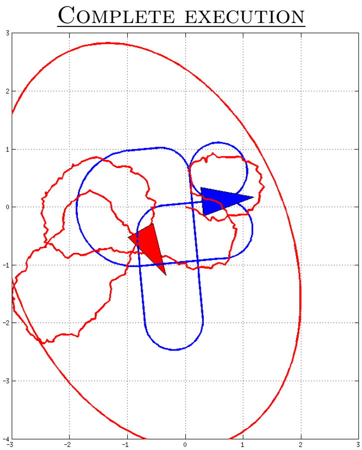
$$\overline{\mu}_t = g(\mu_{t-1}, \mathbf{u}_t, 0)$$

$$\overline{\Sigma}_t = \mathbf{H}_t \Sigma_{t-1} \mathbf{H}_t^T + \mathbf{N}_t \mathbf{R}_t \mathbf{N}_t^T$$

blue: true position and trajectory

red: estimated position and trajectory (μ_t)

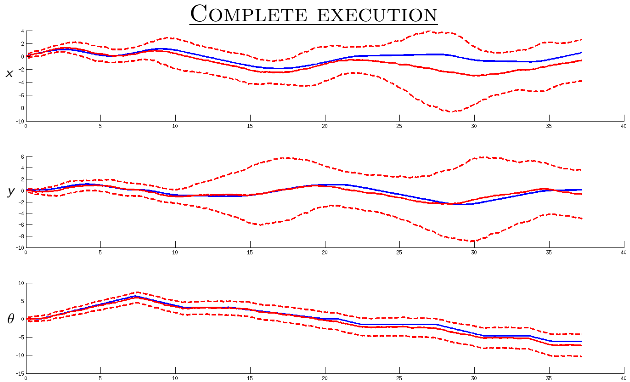
Only Prediction - 3



The complete path with the prediction step

Notice that the true position is inside the ellipse

Only Prediction - Graphs



The complete path with the prediction step

Notice that the true position is inside the ellipse

Prediction - Code

A snapshot of code

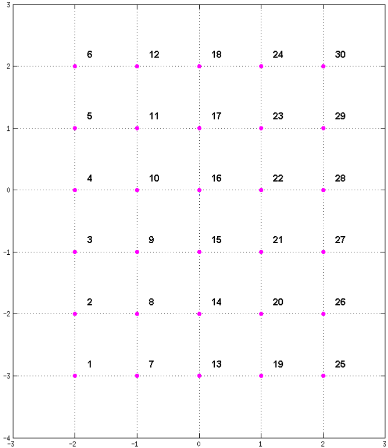
```
G = [ 1, 0, - dy*cos(th) - dx*sin(th);
      0, 1,   dx*cos(th) - dy*sin(th);
      0, 0,                        1];

N = [ cos(th), -sin(th), 0;
      sin(th),  cos(th), 0;
       0,         0,      1];

x1  = cos(th)*(dx) - sin(th)*(dy) + x;
y1  = sin(th)*(dx) + cos(th)*(dy) + y;
th1 = th + dth;

ekf.mu = [x1,y1,th1]';
ekf.sigma = G * ekf.sigma * G' + N * R * N';
```

Outline

Measurement & Update Step - Map

The Map

- $\mathbf{m} : \{\mathbf{p}_1^{(W)}, \mathbf{p}_1^{(W)}, \dots, \mathbf{p}_m^{(W)}\}$
- i.e., a set of points in world coordinates
- Known with *absolute* precision



Measurement & Update Step - The sensor

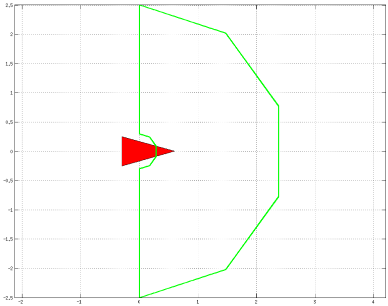
The Sensor

- Measure points in *polar coordinates*

  i.e., ρ, θ values

- w.r.t. robot reference frame
- It recognize the ID of the landmark
  - i.e., Landmarks uniquely identifiable
  - Correspondences are known
  - No data association issues
- Physical limits:
  - Min and max distance
  - Min and max angle
  - Additive zero mean noise on measures

    both for distance and angle

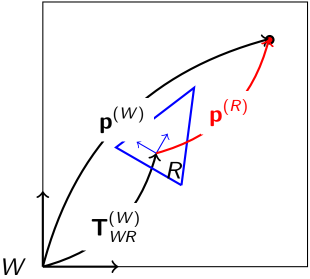


Measurement & Update Step - The equation

Measurement

- $\mathbf{x} = [x, y, \theta]$ is the EKF state

  i.e., the robot complete pose
- Measure: $h_i(\mathbf{x}, \mathbf{p}_i^{(W)}, \delta)$
  - It express what we expect from the sensor
  - Given a single map point $\mathbf{p}_i^{(W)}$
  - Given the estimate robot pose $\mathbf{x} \rightarrow \mathbf{T}_{WR}^{(W)}$
  - i.e., $\mathbf{p}_i^{(R)}$ in polar coordinates wrt



Measurement

- $\mathbf{p}_i^{(R)} = (\mathbf{T}_{WR}^{(W)})^{-1} \mathbf{p}_i^{(W)}$
- $\rho_i = \sqrt{\mathbf{p}_{i_x}^{(R)2} + \mathbf{p}_{i_y}^{(R)2}}$
- $\theta_i = \text{atan2}(\mathbf{p}_{i_y}^{(R)}, \mathbf{p}_{i_x}^{(R)})$

Measurement with noise

- $h_i(\mathbf{x}, \mathbf{p}_i^{(W)}, \delta_i) = \begin{cases} \tilde{\rho}_i = \sqrt{\mathbf{p}_{i_x}^{(R)2} + \mathbf{p}_{i_y}^{(R)2}} + \delta_{\rho_i} \\ \tilde{\theta}_i = \text{atan2}(\mathbf{p}_{i_y}^{(R)}, \mathbf{p}_{i_x}^{(R)}) + \delta_{\theta_i} \end{cases}$
- $\delta_i = [\delta_{\rho_i}, \delta_{\theta_i}]^T \sim \mathcal{N}(0, \mathbf{Q}_i)$

Measurement & Update Step - Jacobians

Measurement equation

$$h_i(\mathbf{x}, \mathbf{p}_i^{(W)}, \delta_i) = \begin{cases} \tilde{\rho}_i = \sqrt{\mathbf{p}_{i_x}^{(R)^2} + \mathbf{p}_{i_y}^{(R)^2}} + \delta_{\rho_i} \\ \tilde{\theta}_i = \text{atan2}(\mathbf{p}_{i_y}^{(R)}, \mathbf{p}_{i_x}^{(R)}) + \delta_{\theta_i} \end{cases}$$

$$\mathbf{p}_i^{(R)} = (\mathbf{T}_{WR}^{(W)})^{-1}\mathbf{p}_i^{(W)}$$

Measurement & Update Step - Jacobians

Measurement equation

$$h_i(\mathbf{x}, \mathbf{p}_i^{(W)}, \delta_i) = \begin{cases} \tilde{\rho}_i = \sqrt{\mathbf{p}_{i_x}^{(R)^2} + \mathbf{p}_{i_y}^{(R)^2}} + \delta_{\rho_i} \\ \tilde{\theta}_i = \text{atan2}(\mathbf{p}_{i_y}^{(R)}, \mathbf{p}_{i_x}^{(R)}) + \delta_{\theta_i} \end{cases}$$

$$\mathbf{p}_i^{(R)} = (\mathbf{T}_{WR}^{(W)})^{-1}\mathbf{p}_i^{(W)}$$

EKF jacobians

- $\mathbf{H}_i = \frac{\partial h_i(\mathbf{x},\mathbf{p},\delta_i)}{\partial \mathbf{x}}\Big|_{\mathbf{x}=\overline{\boldsymbol{\mu}}_{t-1},\mathbf{p}=\mathbf{p}_i^{(W)},\delta_i=0}$
  derivate of the measurement function
  w.r.t. state variables
- $\mathbf{M}_i = \frac{\partial h_i(\mathbf{x},\mathbf{p},\delta_i)}{\partial \delta_i}\Big|_{\mathbf{x}=\overline{\boldsymbol{\mu}}_{t-1},\mathbf{p}=\mathbf{p}_i^{(W)},\delta_i=0}$
  derivate of the measurement function
  w.r.t. noise variables

Measurement & Update Step - Jacobians

Measurement equation

$$h_i(\mathbf{x}, \mathbf{p}_i^{(W)}, \delta_i) = \begin{cases} \tilde{\rho}_i = \sqrt{{\mathbf{p}_{i_x}^{(R)}}^2 + {\mathbf{p}_{i_y}^{(R)}}^2} + \delta_{\rho_i} \\ \tilde{\theta}_i = \text{atan2}(\mathbf{p}_{i_y}^{(R)}, \mathbf{p}_{i_x}^{(R)}) + \delta_{\theta_i} \end{cases}$$

$$\mathbf{p}_i^{(R)} = (\mathbf{T}_{WR}^{(W)})^{-1}\mathbf{p}_i^{(W)}$$

EKF jacobians

- $\mathbf{H}_i = \frac{\partial h_i(\mathbf{x},\mathbf{p},\delta_i)}{\partial \mathbf{x}}\Big|_{\mathbf{x}=\overline{\boldsymbol{\mu}}_{t-1},\mathbf{p}=\mathbf{p}_i^{(W)},\delta_i=0}$
  derivate of the measurement function w.r.t. state variables
- $\mathbf{M}_i = \frac{\partial h_i(\mathbf{x},\mathbf{p},\delta_i)}{\partial \delta_i}\Big|_{\mathbf{x}=\overline{\boldsymbol{\mu}}_{t-1},\mathbf{p}=\mathbf{p}_i^{(W)},\delta_i=0}$
  derivate of the measurement function w.r.t. noise variables

Jacobians

- $\mathbf{H}_i =$

$$\begin{bmatrix} \frac{\partial h_{i1}(\mathbf{x},\mathbf{p},\delta)}{\partial x} & \frac{\partial h_{i1}(\mathbf{x},\mathbf{p},\delta)}{\partial y} & \frac{\partial h_{i1}(\mathbf{x},\mathbf{p},\delta)}{\partial \theta} \\ \frac{\partial h_{i2}(\mathbf{x},\mathbf{p},\delta)}{\partial x} & \frac{\partial h_{i2}(\mathbf{x},\mathbf{p},\delta)}{\partial y} & \frac{\partial h_{i2}(\mathbf{x},\mathbf{p},\delta)}{\partial \theta} \end{bmatrix}$$

$$= \dots$$

- $\mathbf{M}_i = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Measurement & Update Step - Jacobians

Measurement equation

$$h_i(\mathbf{x}, \mathbf{p}_i^{(W)}, \delta_i) = \begin{cases} \tilde{\rho}_i = \sqrt{\mathbf{p}_{i_x}^{(R)2} + \mathbf{p}_{i_y}^{(R)2}} + \delta_{\rho_i} \\ \tilde{\theta}_i = \text{atan2}(\mathbf{p}_{i_y}^{(R)}, \mathbf{p}_{i_x}^{(R)}) + \delta_{\theta_i} \end{cases}$$

$$\mathbf{p}_i^{(R)} = (\mathbf{T}_{WR}^{(W)})^{-1}\mathbf{p}_i^{(W)}$$

EKF jacobians

- $\mathbf{H}_i = \frac{\partial h_i(\mathbf{x},\mathbf{p},\delta_i)}{\partial \mathbf{x}}\Big|_{\mathbf{x}=\overline{\boldsymbol{\mu}}_{t-1},\mathbf{p}=\mathbf{p}_i^{(W)},\delta_i=0}$
  derivate of the measurement function w.r.t. state variables
- $\mathbf{M}_i = \frac{\partial h_i(\mathbf{x},\mathbf{p},\delta_i)}{\partial \delta_i}\Big|_{\mathbf{x}=\overline{\boldsymbol{\mu}}_{t-1},\mathbf{p}=\mathbf{p}_i^{(W)},\delta_i=0}$
  derivate of the measurement function w.r.t. noise variables

Jacobians

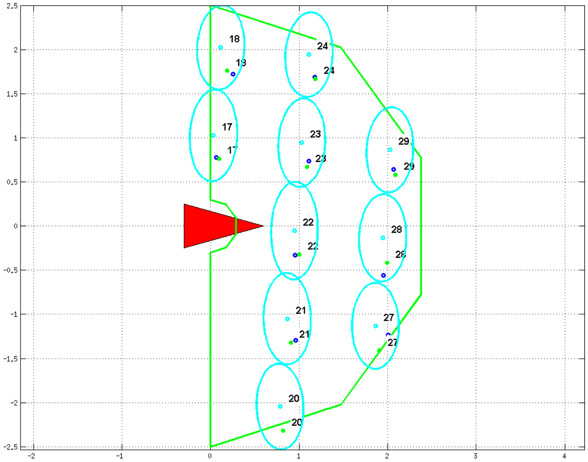
- $\mathbf{H}_i =$
  $$\begin{bmatrix} \frac{\partial h_{i1}(\mathbf{x},\mathbf{p},\delta)}{\partial x} & \frac{\partial h_{i1}(\mathbf{x},\mathbf{p},\delta)}{\partial y} & \frac{\partial h_{i1}(\mathbf{x},\mathbf{p},\delta)}{\partial \theta} \\ \frac{\partial h_{i2}(\mathbf{x},\mathbf{p},\delta)}{\partial x} & \frac{\partial h_{i2}(\mathbf{x},\mathbf{p},\delta)}{\partial y} & \frac{\partial h_{i2}(\mathbf{x},\mathbf{p},\delta)}{\partial \theta} \end{bmatrix}$$
  $= \ldots$
- $\mathbf{M}_i = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Some notes

- Not all $h_i(\cdot)$ are *valid*
  e.g., $\rho_i \notin [\rho_{min}, \rho_{max}]$
  e.g., $\theta_i \notin [\theta_{min}, \theta_{max}]$
- We select a subset of $h_i(\cdot)$

Measurement & Update Step - Measurement Details

Measurement in robot frame



- Cyan: the *predicted measure*, $h_i(\cdot)$
- Green: the real map point in robot coordinates
- Blue: the noisy sensor measurement $\mathbf{z}_i$
- Ellipses: given by covariance $\mathbf{S}_t = \mathbf{H}_t\overline{\Sigma}_t\mathbf{H}_t^T + \mathbf{M}_t\mathbf{Q}_t\mathbf{M}_t^T$
- Innovation: $\mathbf{z}_i - h_i(\cdot)$

Measurement & Update Step - Unique Update

THE MEASUREMENTS

- $h_i(\cdot)$, $z_i(\cdot)$, $\mathbf{H}_i$, $\mathbf{M}_i(\cdot)$, $\mathbf{Q}_i(\cdot)$
  feasible measurements and Jacobians
- How to update?

THE COMPLETE MEASUREMENTS

- $h(\mathbf{x}, \mathbf{m}, \delta) = \begin{cases} h_1(\mathbf{x}, \mathbf{p}_1^{(W)}, \delta_1) \\ h_2(\mathbf{x}, \mathbf{p}_2^{(W)}, \delta_2) \\ \cdots \\ h_m(\mathbf{x}, \mathbf{p}_m^{(W)}, \delta_m) \end{cases}$
- $\delta = \begin{bmatrix} \delta_1^T & \delta_2^T & \cdots & \delta_m^T \end{bmatrix}^T$

$$\mathbf{H} = \begin{bmatrix} \frac{\partial h_1(\mathbf{x}, \mathbf{p}_1^{(W)}, \delta_1)}{\partial \mathbf{x}} \\ \frac{\partial h_2(\mathbf{x}, \mathbf{p}_2^{(W)}, \delta_2)}{\partial \mathbf{x}} \\ \cdots \\ \frac{\partial h_m(\mathbf{x}, \mathbf{p}_m^{(W)}, \delta_m)}{\partial \mathbf{x}} \end{bmatrix} \quad \mathbf{M} = \begin{bmatrix} \frac{\partial h_1(\mathbf{x}, \mathbf{p}_1^{(W)}, \delta_1)}{\partial \delta_\mathbf{1}} \\ \frac{\partial h_2(\mathbf{x}, \mathbf{p}_2^{(W)}, \delta_2)}{\partial \delta_\mathbf{2}} \\ \cdots \\ \frac{\partial h_m(\mathbf{x}, \mathbf{p}_m^{(W)}, \delta_m)}{\partial \delta_\mathbf{m}} \end{bmatrix}$$

Measurement & Update Step - Unique Update

The measurements

- $h_i(\cdot)$, $z_i(\cdot)$, $\mathbf{H}_i$, $\mathbf{M}_i(\cdot)$, $\mathbf{Q}_i(\cdot)$
  feasible measurements and Jacobians
- How to update?

The complete measurements

- $$h(\mathbf{x}, \mathbf{m}, \delta) = \begin{cases} h_1(\mathbf{x}, \mathbf{p}_1^{(W)}, \delta_1) \\ h_2(\mathbf{x}, \mathbf{p}_2^{(W)}, \delta_2) \\ \cdots \\ h_m(\mathbf{x}, \mathbf{p}_m^{(W)}, \delta_m) \end{cases}$$
- $\delta = \begin{bmatrix} \delta_1^T & \delta_2^T & \cdots & \delta_m^T \end{bmatrix}^T$

The update

$$\mathbf{h} = \begin{bmatrix} h_1^T & h_2^T & \cdots & h_m^T \end{bmatrix}^T$$

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_1^T & \mathbf{H}_2^T & \cdots & \mathbf{H}_m^T \end{bmatrix}^T$$

$$\mathbf{z} = \begin{bmatrix} z_1^T & z_2^T & \cdots & z_m^T \end{bmatrix}^T$$

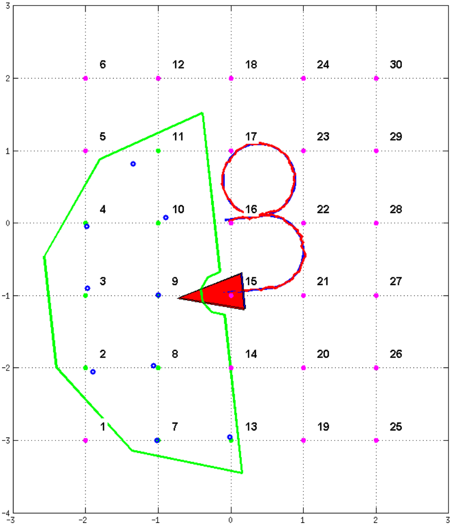
$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{M}_2 & \cdots & 0 \\ & \cdots & \cdots & \\ 0 & \cdots & \mathbf{M}_{m-1} & 0 \\ & \cdots & \cdots & \\ 0 & \cdots & 0 & \mathbf{M}_m \end{bmatrix}$$

$$\mathbf{H} = \begin{bmatrix} \frac{\partial h_1(\mathbf{x}, \mathbf{p}_1^{(W)}, \delta_1)}{\partial \mathbf{x}} \\ \frac{\partial h_2(\mathbf{x}, \mathbf{p}_2^{(W)}, \delta_2)}{\partial \mathbf{x}} \\ \cdots \\ \frac{\partial h_m(\mathbf{x}, \mathbf{p}_m^{(W)}, \delta_m)}{\partial \mathbf{x}} \end{bmatrix} \quad \mathbf{M} = \begin{bmatrix} \frac{\partial h_1(\mathbf{x}, \mathbf{p}_1^{(W)}, \delta_1)}{\partial \delta_1} \\ \frac{\partial h_2(\mathbf{x}, \mathbf{p}_2^{(W)}, \delta_2)}{\partial \delta_2} \\ \cdots \\ \frac{\partial h_m(\mathbf{x}, \mathbf{p}_m^{(W)}, \delta_m)}{\partial \delta_m} \end{bmatrix}$$

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{Q}_2 & \cdots & 0 \\ & \cdots & \cdots & \\ 0 & \cdots & & 0 \\ & \cdots & \mathbf{Q}_{m-1} & \\ 0 & \cdots & 0 & \mathbf{Q}_m \end{bmatrix}$$

Measurement & Update Step - Some Steps - 1

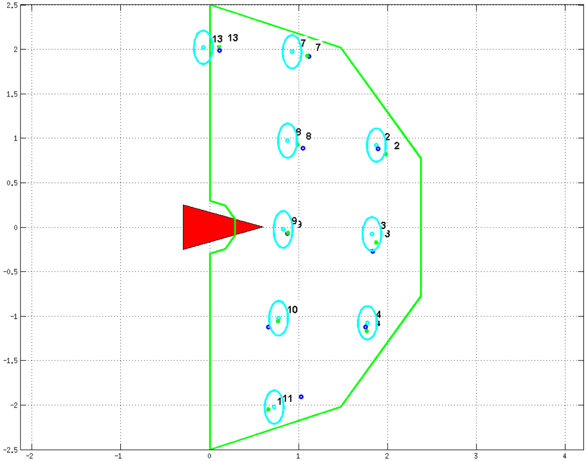
In the world



Covariance on robot pose is reduced

Measurement & Update Step - Some Steps - 2

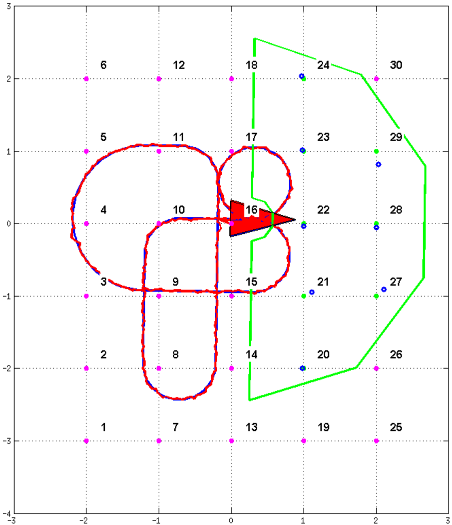
In the robot



Covariance on measurement is reduced due to the minor uncertainty in the pose

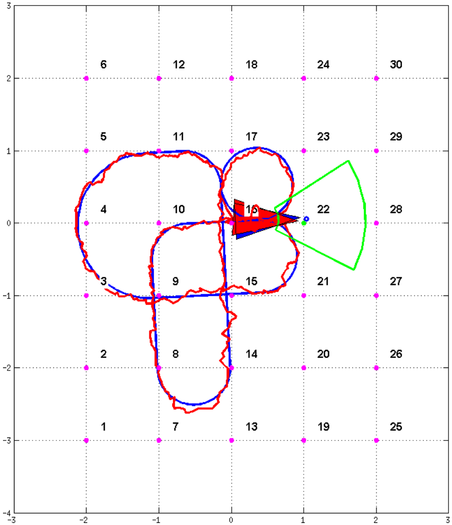
Measurement & Update Step - Complete Execution

The complete path



Measurement & Update Step - Worse Sensor

The complete path - Worse Sensor



Estimated trajectory is less precise, covariance on robot pose is bigger

Outline

Correspondences

Correspondences

- Correspondences are known → this is uncommon in real environments
- If correspondences are unknown we have to perform the *data association*

Data association

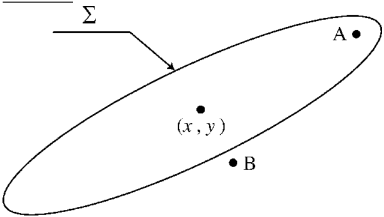
- Given a set of measurements $\{\mathbf{z}_i\}, i = 1 : m$
- Given a set of *measurements prediction* $\{\mathbf{h}_j\}, j = 1 : w$
- We have to select correspondences c_{ij}

"Dummy" approach

1. $k = 1$
2. Select w such that $\mathbf{z}_w$ closest to $\mathbf{h}_k$
3. Remove $\mathbf{z}_w$ from $\{\mathbf{z}_i\}$
4. Repeat from 2

Mahalanobis Distance

GIVEN



- Given A, B coordinates
- Distance to (x, y)
- Suppose to know covariance Σ
- i.e., $\sim \mathcal{N}(\mu = [x, y], \Sigma)$

EUCLIDEAN DISTANCE

- A is closest to x, y
- B is far

MAHALANOBIS DISTANCE

- $D^2 = (x - \mu)^T \Sigma^{-1} (x - \mu)$
- Squared distance weighted for the inverse of covariance
- $D^2(A) < D^2(B)$,
  A is inside the covariance ellipse
- It is a scaled and rotated distance
- Same probability $=$ same distance
- D^2 is distributed as a $\chi^2(n)$

Data association

Data association with D^2

1. $k = 1$
2. Select w such that $\mathbf{z}_w$ closest to $\mathbf{h}_k$ in $D^2(\mathbf{z}_w, \mathbf{h}_k)$
3. Remove $\mathbf{z}_w$ from $\{\mathbf{z}_i\}$
4. Repeat from 2

- Further, when minimum distance is too high, stop association algorithm

Is it the right approach?

- Is better than the Euclidean distance data association
- Could performs wrong associations
- It consider only *individual compatibility*
- Resulting in a bad localization
- Better approach will consider *joint compatibility* or performs *multi hypothesis*

Outline

Towards non parametric filters

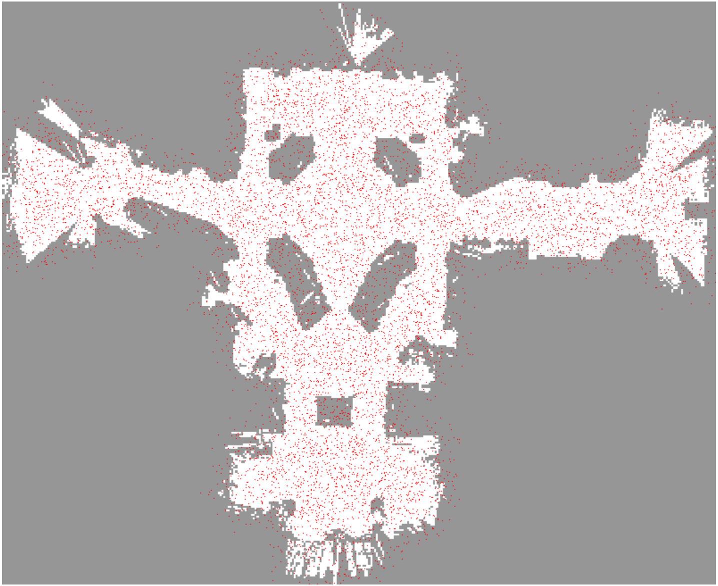
EKF summary

- Efficient: $< O(n^3)$
- Not optimal, suffers of linearizations
- Can diverge with huge nonlinearities
- Works surprisingly well even when all assumptions are violated
- Not suitable for global localization or kidnapped robot problem

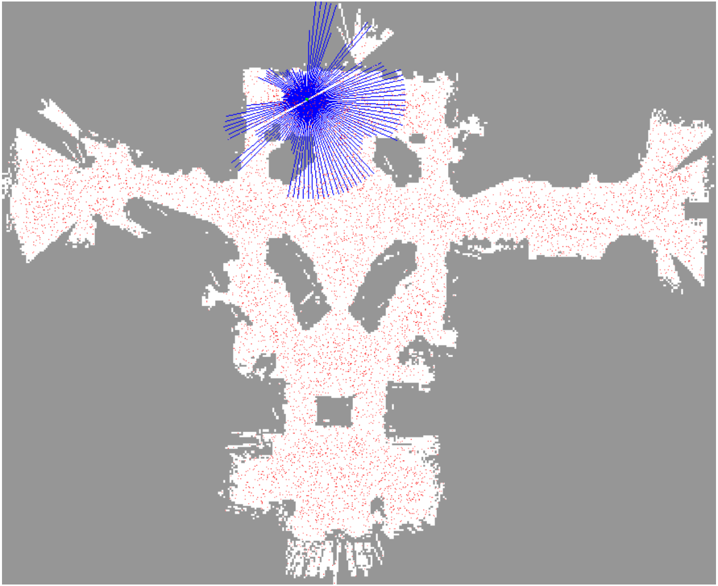
Particle filter

- Represents belief by *random samples* from distributions
- Can treat nonlinear and non gaussian problems
- a.k.a. *Sequential Monte Carlo* (SMC) method
- Performs Monte Carlo Localization (MCL)

MCL - 1



MCL - 2



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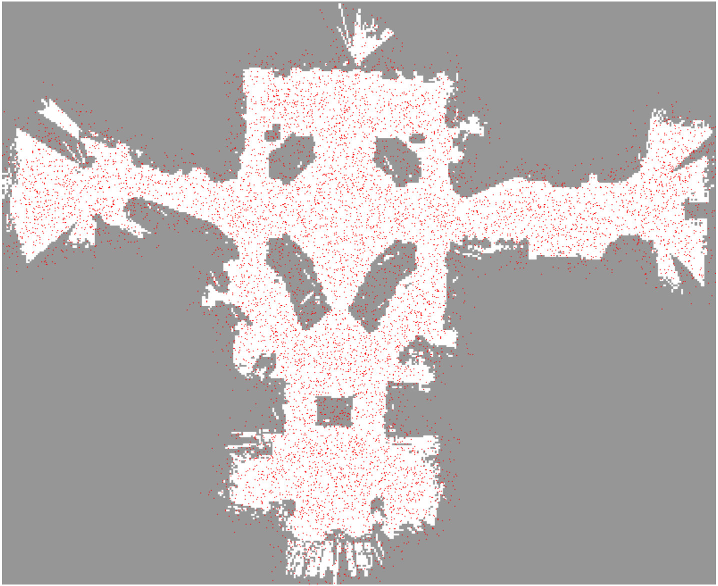

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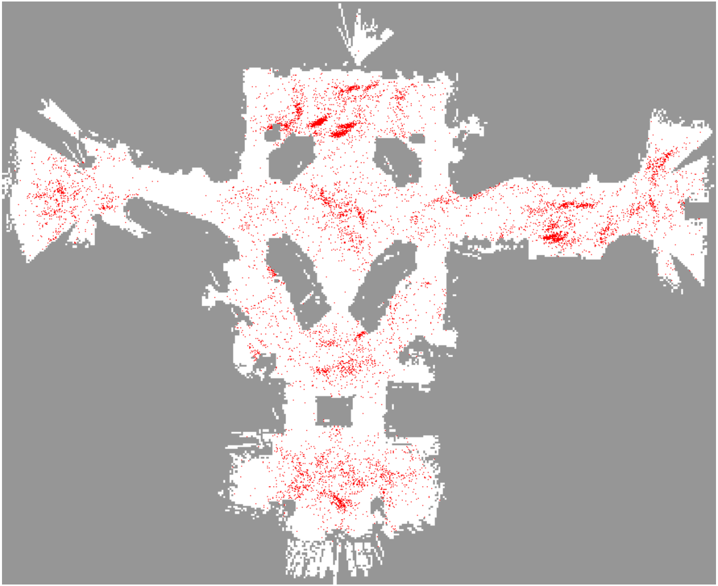

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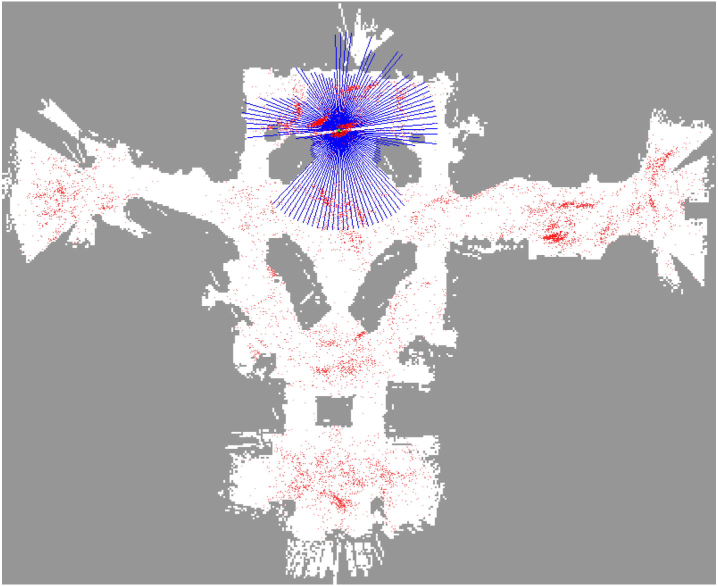

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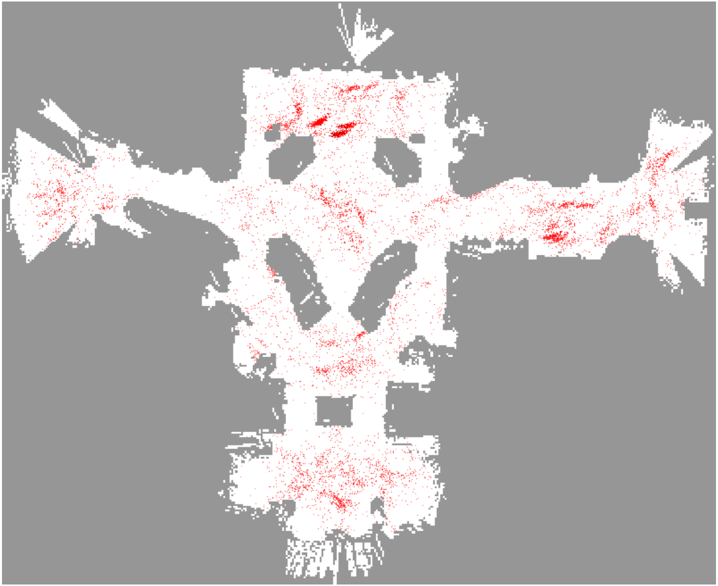

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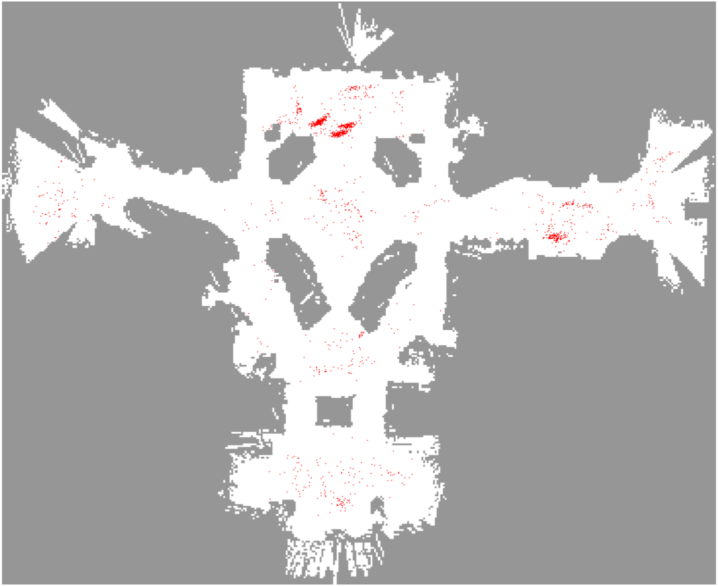
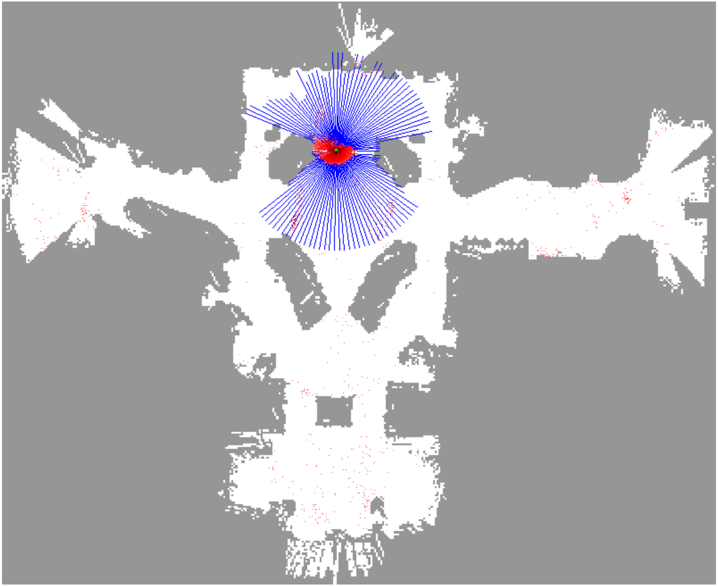


MCL - 8

MCL - 9



MCL - 10

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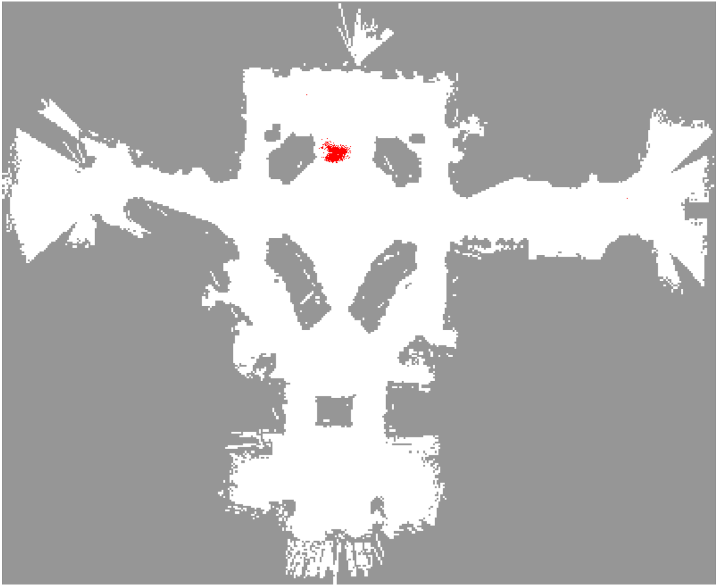

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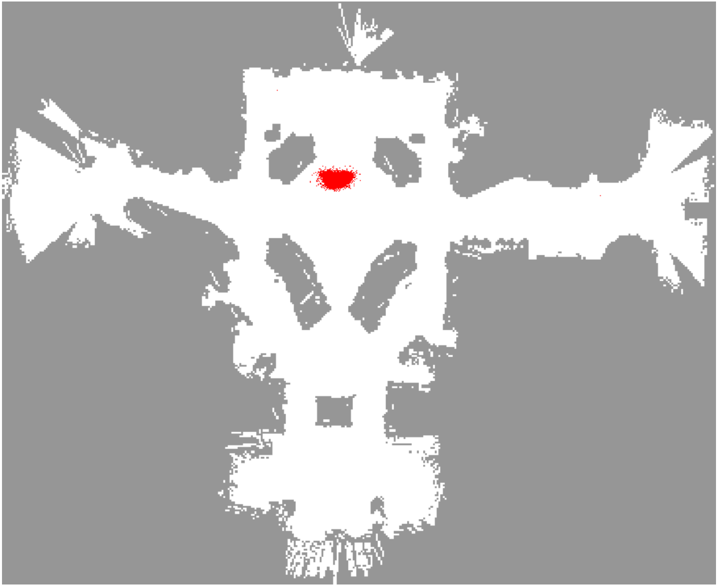

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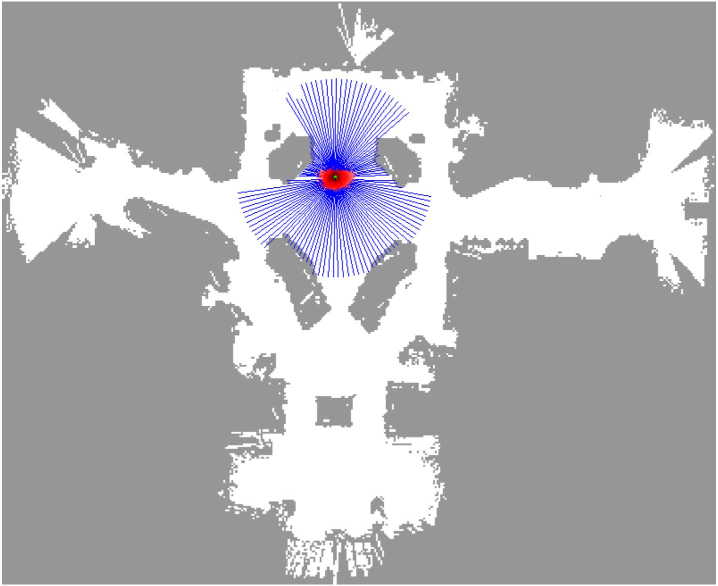

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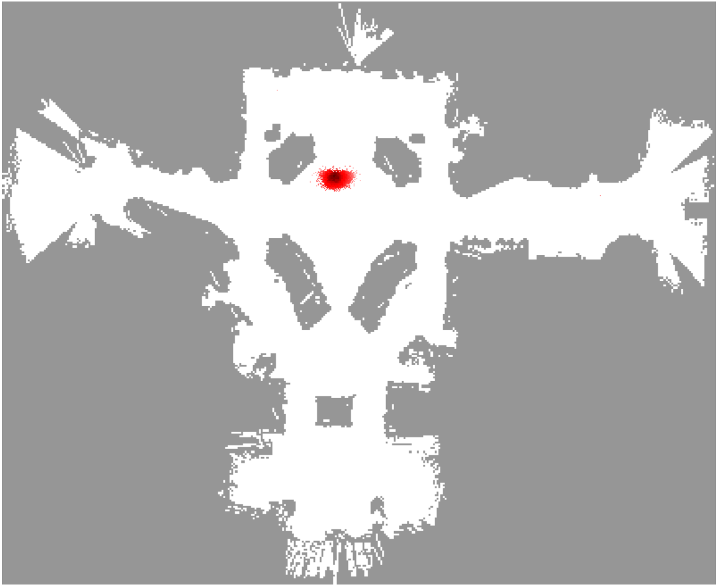

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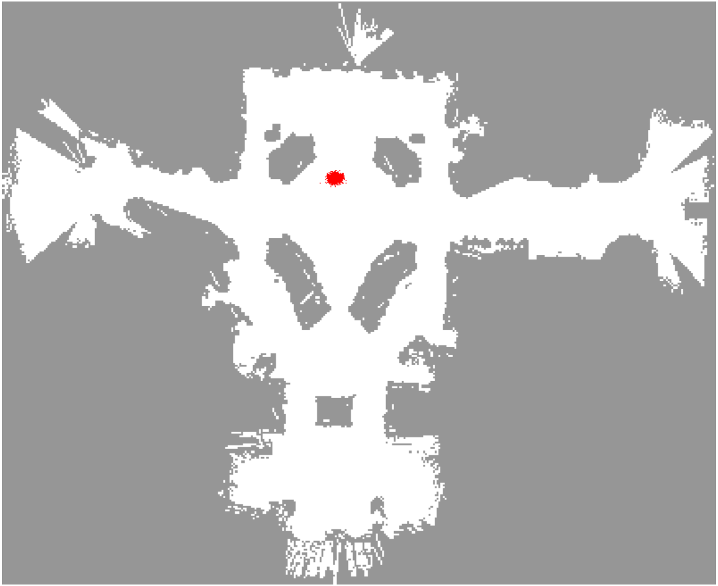

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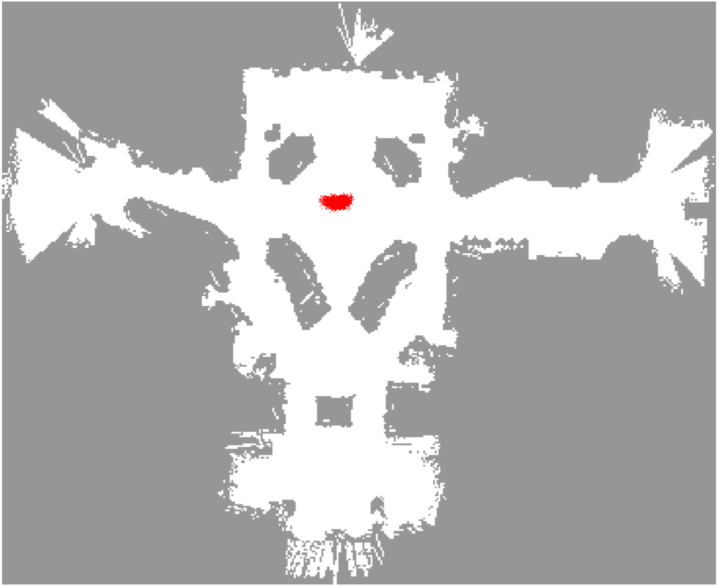

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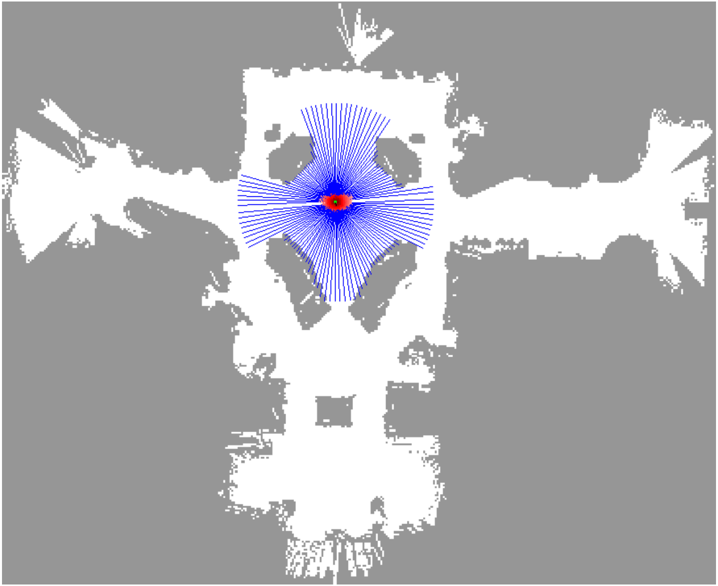

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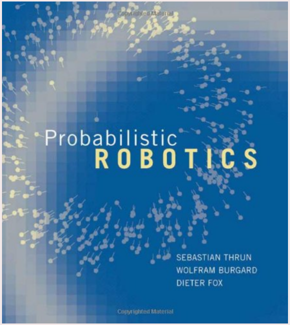

Chapters 2, 3, 4, 7.