

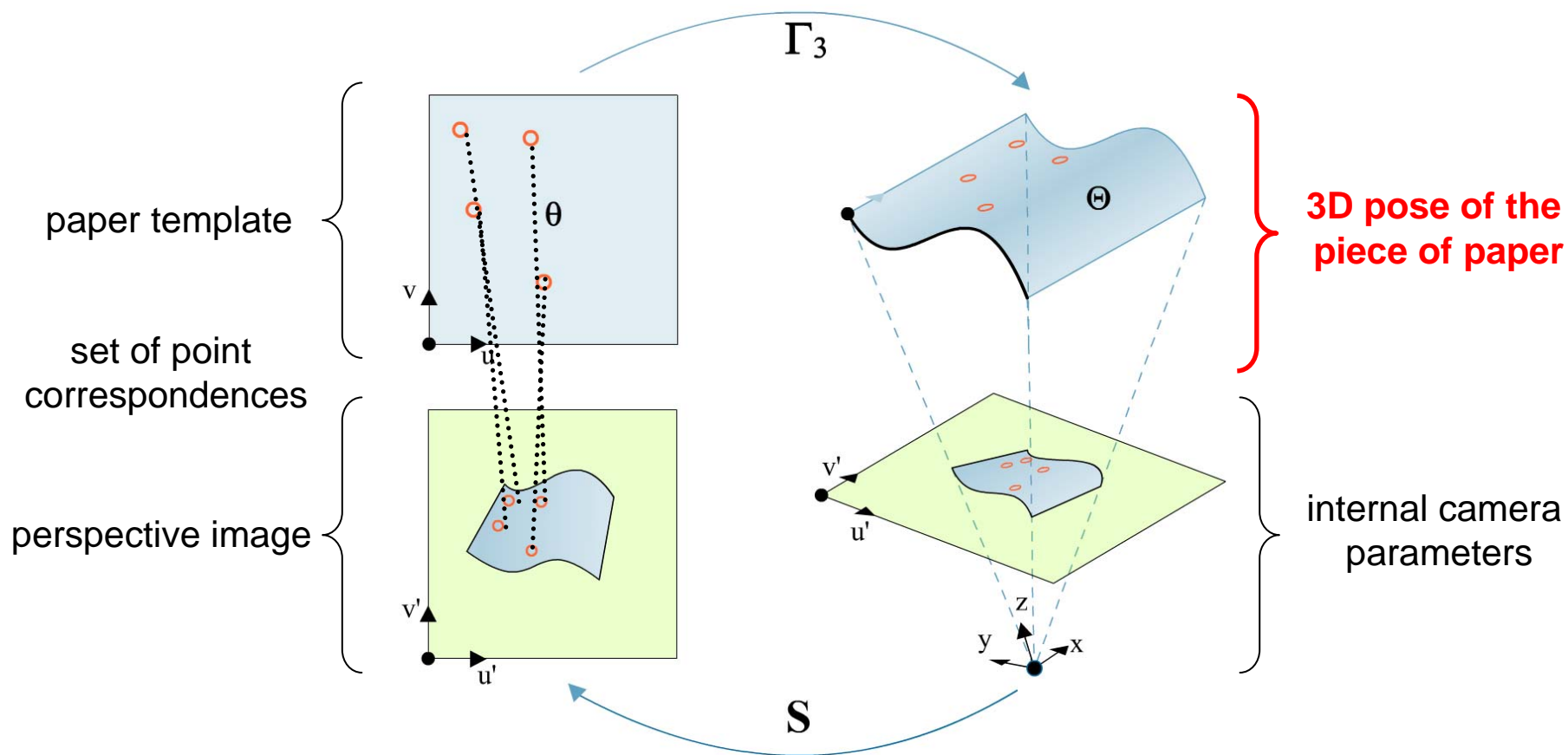
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## Template-based Paper Reconstruction from a Single Image is Well Posed when the Rulings are Parallel

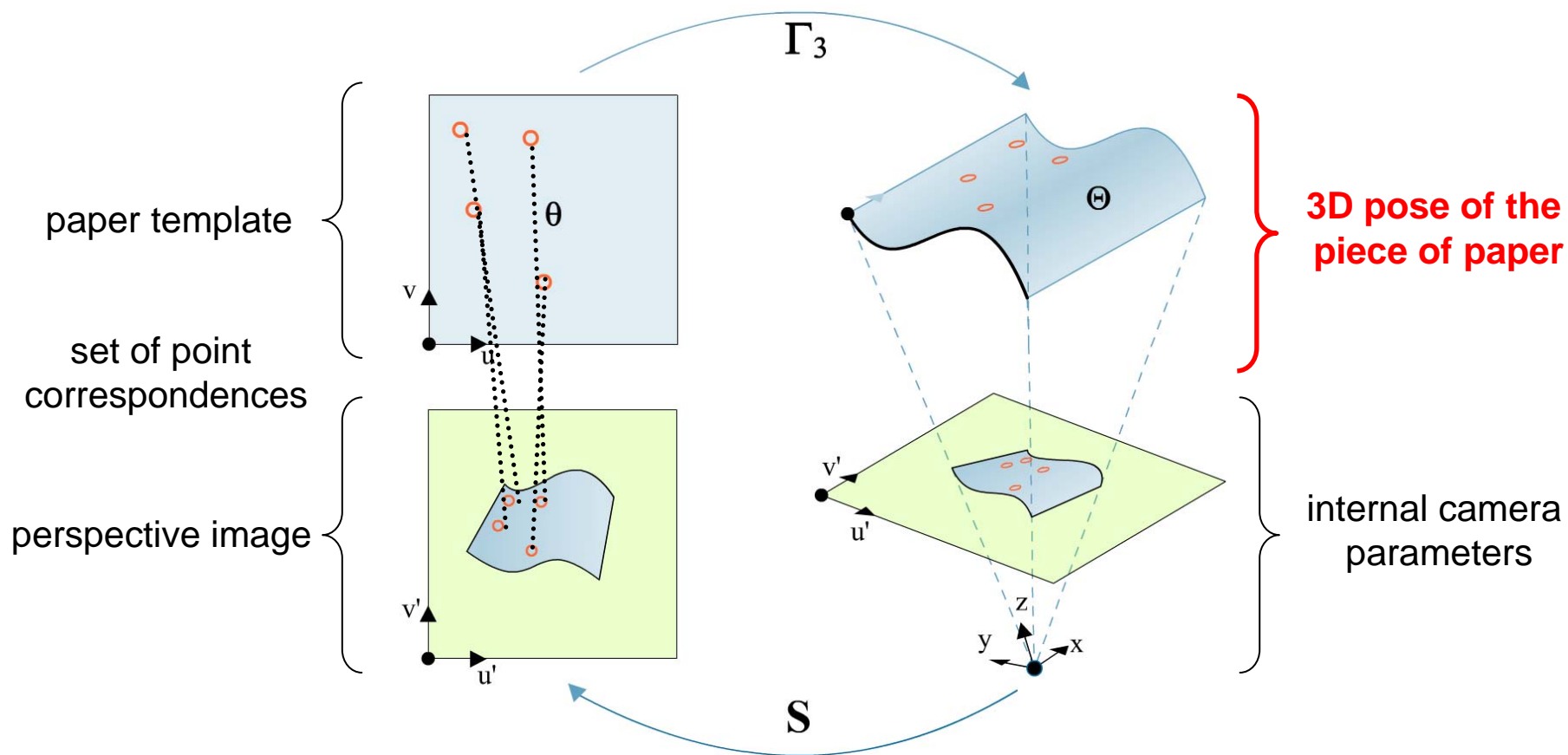
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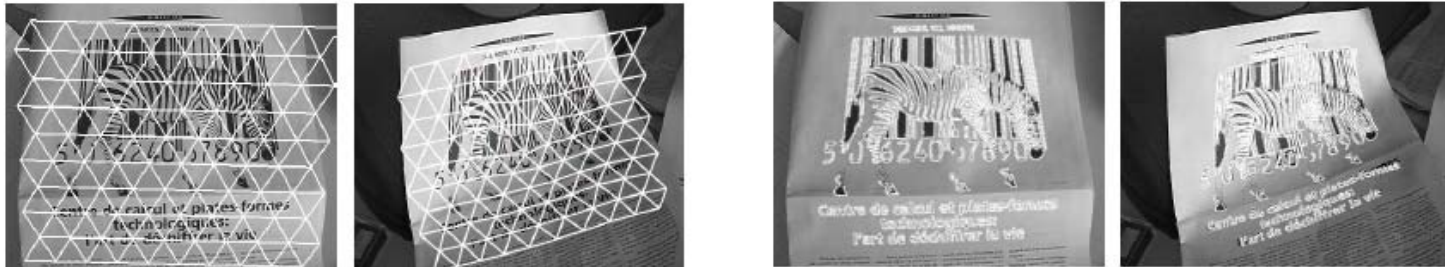
We aim to reconstruct the pose of a piece of paper which is subject to a subset of possible isometries



We show that for particular isometries this is a well posed problem



- Template-based monocular deformable surface registration may be performed using general models
  - Generic deformable surfaces using triangular mesh grids  
(Julien Pilet, Vincent Lepetit, Pascal Fua)



- Monocular deformable surface reconstruction is possible if some priors are known
  - 3D Morphable Models for face reconstruction  
(Volker Blanz and Thomas Vetter)



→ Great works  
to describe real deformations

or learnt models



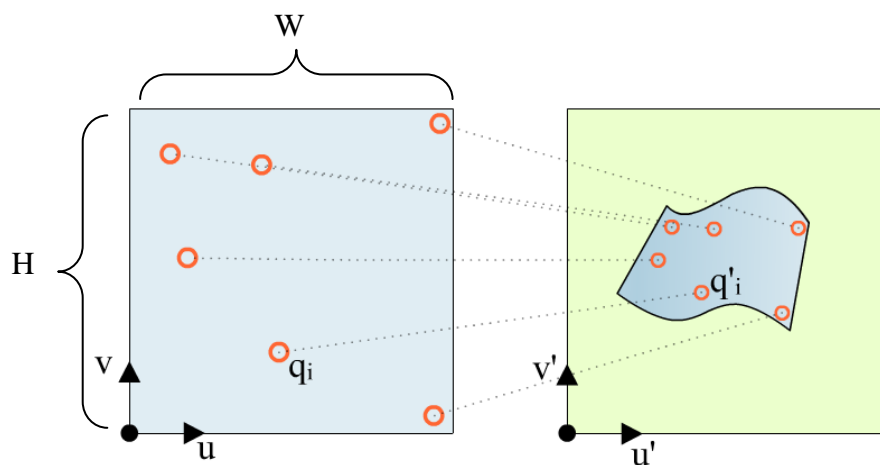
- We address the case of developable surface to model material such as paper
  - Useful for augmentation
- Paper reconstruction may be performed using shape-from-contour
  - mainly for document digitization
    - requires the full knowledge of the contours
  - Not useful in the case of occlusion



**a well-posed problem**

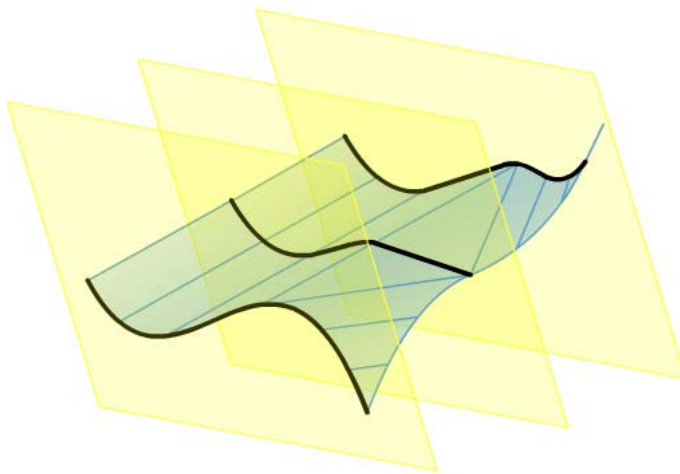


In order to perform a full 3D reconstruction we assume:

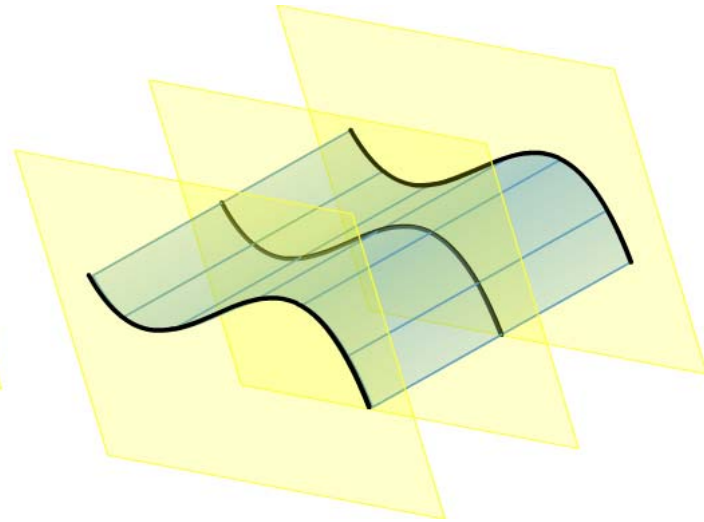


- Set of point correspondences  $\{q_i\} \rightarrow \{q'_i\}$
- Internal camera parameters known  $\mathbf{S}$
- Metric size of the template  $(W, H)$
- Physical model, developable surfaces
  - Deformations are **isometries**, thus distances are maintained
  - Vanishing gaussian curvatur

- The general case of isometric deformations is ill-posed
- We consider a subset of the possible isometries
  - The rulings of the developable surface are constrained to be parallel, i.e. the surface is a **generalized cylinder**
  - Intuitively this is what happens when book pages are deformed by keeping the binding and the opposite edge parallel.



Generic isometry, ill posed



Generalized cylinder, well posed



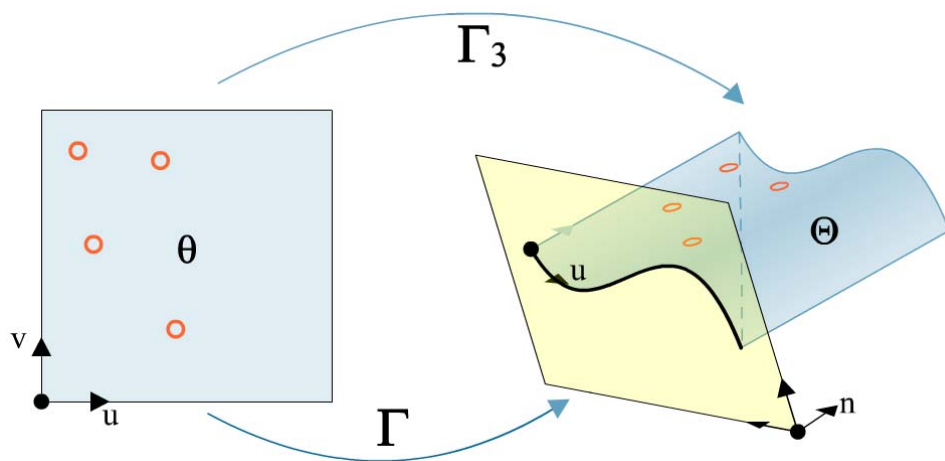


## reduction to a 2D problem



- In the case of a generalized cylinder the surface is parameterized as follows:
  - A generatrix plane  $\pi$  which is perpendicular to all rulings and contains the lower border of the surface
  - A transformation  $T$  which maps the  $XY$  plane to  $\pi$ , the origin to the bottom left corner, the  $X$  axis to the corner segment
  - A  $\mathbb{R} \rightarrow \mathbb{R}^2$  mapping  $\Gamma$  which maps  $u$  coordinates to a 2D curve on  $\pi$

$$\Gamma_3(u, v) = T \begin{pmatrix} \Gamma(u) \\ v \\ 1 \end{pmatrix} \quad \forall u, v \in \theta$$





- By considering the projection equation we can derive that:

$$q' \sim S \cdot T \begin{pmatrix} \Gamma(u) \\ v \\ 1 \end{pmatrix} \quad \begin{pmatrix} \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{pmatrix} \cdot \begin{pmatrix} \square \\ \square \\ \square \\ \square \end{pmatrix}$$

$$q' \sim (s_1 \quad s_2 \quad (s_4 + s_3v)) \cdot \begin{pmatrix} \Gamma(u) \\ 1 \end{pmatrix}$$

$$q' \sim \begin{pmatrix} S_{va} \\ S_{vb} \\ S_{vc} \end{pmatrix} \cdot \begin{pmatrix} \Gamma(u) \\ 1 \end{pmatrix} \quad \begin{pmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{pmatrix} \cdot \begin{pmatrix} \square \\ \square \\ \square \end{pmatrix}$$

$$q'_u \sim \begin{pmatrix} S_{va} \\ S_{vc} \end{pmatrix} \cdot \begin{pmatrix} \Gamma(u) \\ 1 \end{pmatrix} \quad \begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix} \cdot \begin{pmatrix} \square \\ \square \end{pmatrix}$$

$$q'_v \sim \begin{pmatrix} S_{vb} \\ S_{vc} \end{pmatrix} \cdot \begin{pmatrix} \Gamma(u) \\ 1 \end{pmatrix} \quad \begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix} \cdot \begin{pmatrix} \square \\ \square \end{pmatrix}$$



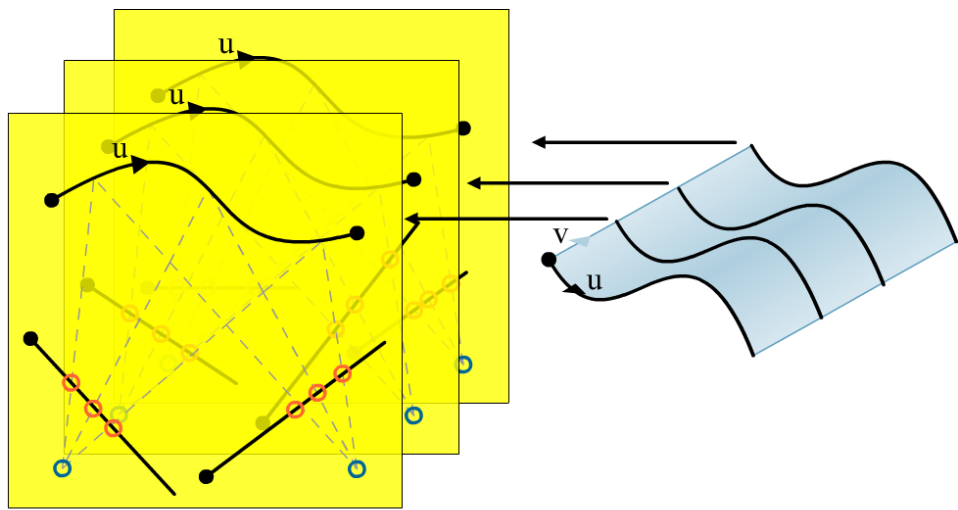
$$q' \sim S \cdot T \begin{pmatrix} \Gamma(u) \\ v \\ 1 \end{pmatrix}$$



$$q'_u \sim \begin{pmatrix} S_{va} \\ S_{vc} \end{pmatrix} \cdot \begin{pmatrix} \Gamma(u) \\ 1 \end{pmatrix}$$

$$q'_v \sim \begin{pmatrix} S_{vb} \\ S_{vc} \end{pmatrix} \cdot \begin{pmatrix} \Gamma(u) \\ 1 \end{pmatrix}$$

- The problem is equivalent to the reconstruction of 2D points given a pair of 1D cameras for each surface slice
- $u$  varies the point position over the 2D curve
- $v$  varies the two cameras internal parameters



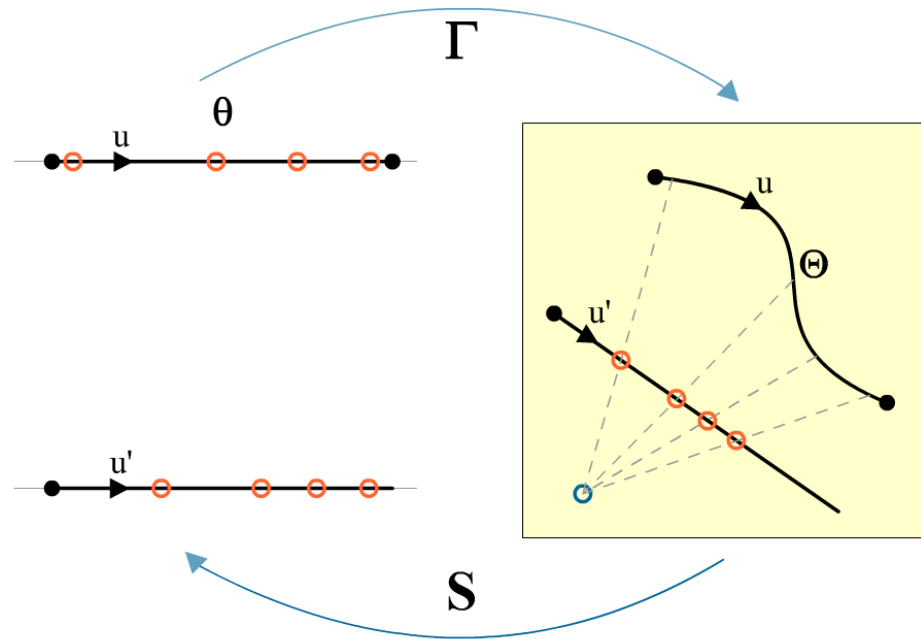


## **solving the problem**

- Isometries preserves gaussian curvature:  
the gaussian curvature is, thus, vanishing everywhere
  - since the parameterization is given by a developable surface  
this constraint is enforced by construction
- Isometries preserves the metric:
  - By construction distances are preserved along the rulings
  - Since we are assuming a generalized cylinder, if the metric is preserved on  $\pi$  section then it is preserved everywhere

→  $\Gamma$  must be a 1D isometry

- Metric constraints:  $\|\Gamma_u\|^2 = 1 \quad \forall u \in \Theta$



- Moreover, we aim to minimize:
  - the reprojection error of the point correspondences
  - a smoothing term

- The problem is expressed as a functional optimization:

$$\arg \min_{\Gamma} (E_d[\Gamma] + E_m[\Gamma_u] + E_s[\Gamma_{uu}]) = \arg \min_{\Gamma} \left( \int e(u, \Gamma, \Gamma_u, \Gamma_{uu}) \right)$$

- This problem depends on the free variable  $\mathbf{u}$ , function  $\Gamma$  and its **first and second derivatives**
- The problem possess natural boundary condition (i.e. the boundary are not fixed)
- The functional  $\mathbf{E}$  is given by the weighted sum of:
  - $E_d[\Gamma]$  : data term, which describe the reprojection error
  - $E_s[\Gamma_{uu}] = \int_{\theta} \|\Gamma_{uu}(u)\|^2$  : smoothing term
  - $E_m[\Gamma_u] = \int_{\theta} \left( \|\Gamma_u(u)\|^2 - 1 \right)^2$  : metric term



$$\Gamma = \arg \min_{\Gamma} \left( \int e(u, \Gamma, \Gamma_u, \Gamma_{uu}) \right)$$

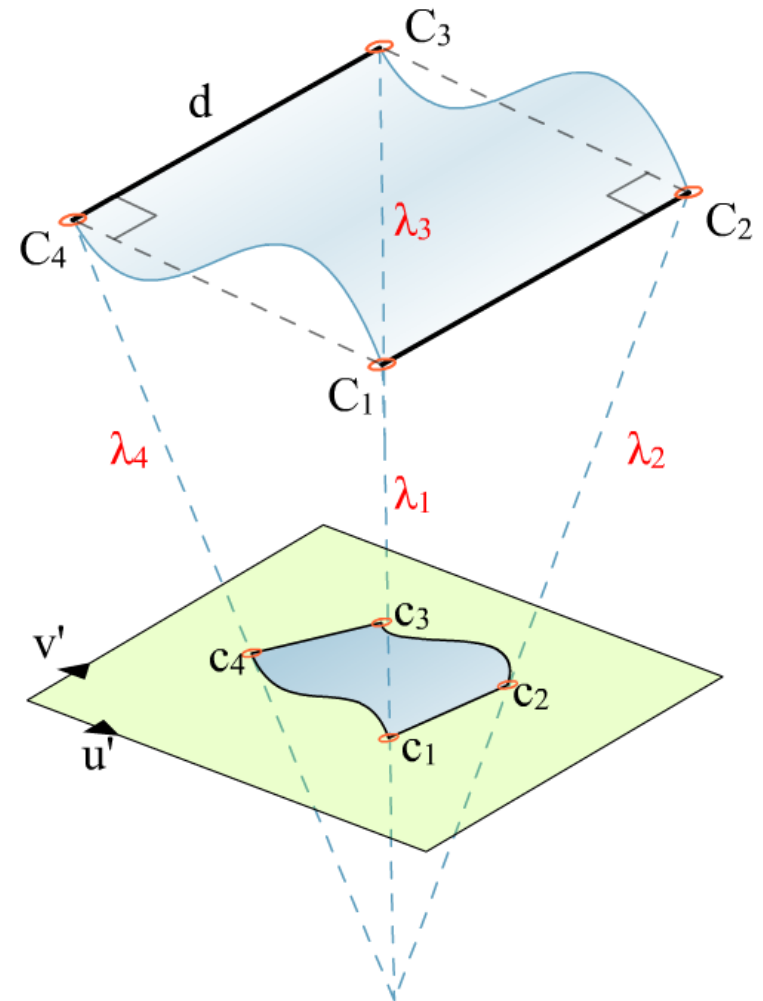
- This functional optimization is solved by applying the Euler-Lagrange equations
  - this gives a system of PDEs depending up to the fourth derivatives of  $\Gamma$  and a set of PDEs related to the natural boundary condition
- The PDE system is solved using numerical methods:
  - The domain is sampled at  $N$  nodes
  - Derivatives are replaced by finite differences approximation



**recover the generatrix plane**

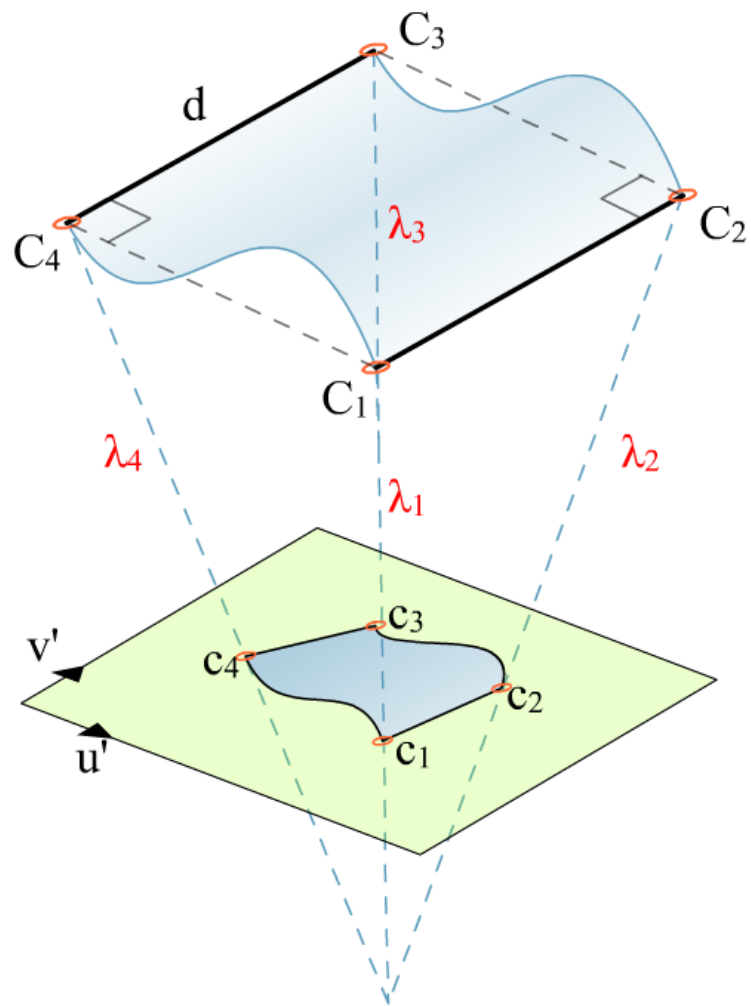
- To exploit the exposed parametrization the generatrix plane transformation is needed
- This can be done by exploiting
  - the template dimensions
  - at least two pair of points on the same ruling, for instance the corners of the largest visible rectangle
- Using the template dimension the points distances are easily calculated
- We know the camera internal parameter
- The problem can be solved using an optimization procedure

- Done exploiting the template dimensions and at least two pairs of points on the same ruling
- Using the template dimension the inter point distances are easily calculated
- The problem can be solved using an optimization procedure



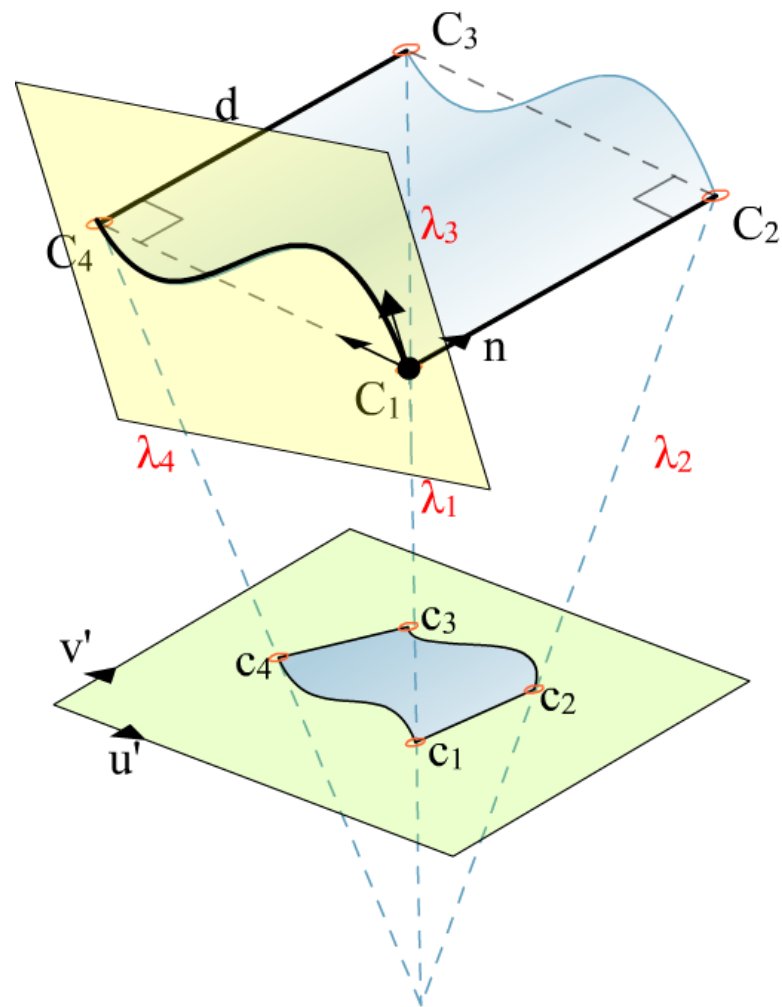
## Generatrix plane recovery (2)

- Given the four points  $c_1, c_2$  and  $c_3, c_4$ , the two segments length  $d$  and the camera internal parameters
- The unknowns are the four perspective depths  $\lambda_1, \lambda_2, \lambda_3$  and  $\lambda_4$
- These may be recovered by enforcing the constraints:
  - $C_1C_2$  is parallel to  $C_3C_4$ ,
  - $C_1C_2$  is orthogonal to  $C_1C_4$ ,
  - $C_1C_2$  has length  $d$ ,
  - $C_3C_4$  has length  $d$



## Generatrix plane recovery (3)

- The generatrix plane  $\pi$  is orthogonal to the plane containing the detected rectangle
- In particular we consider a transformation  $\mathbf{T}$  which brings
  - The plane  $XY$  to  $\pi$ ,
  - The axis  $X$  parallel to  $C_1C_4$ ,
  - The axis  $Z$  parallel to  $C_1C_2$

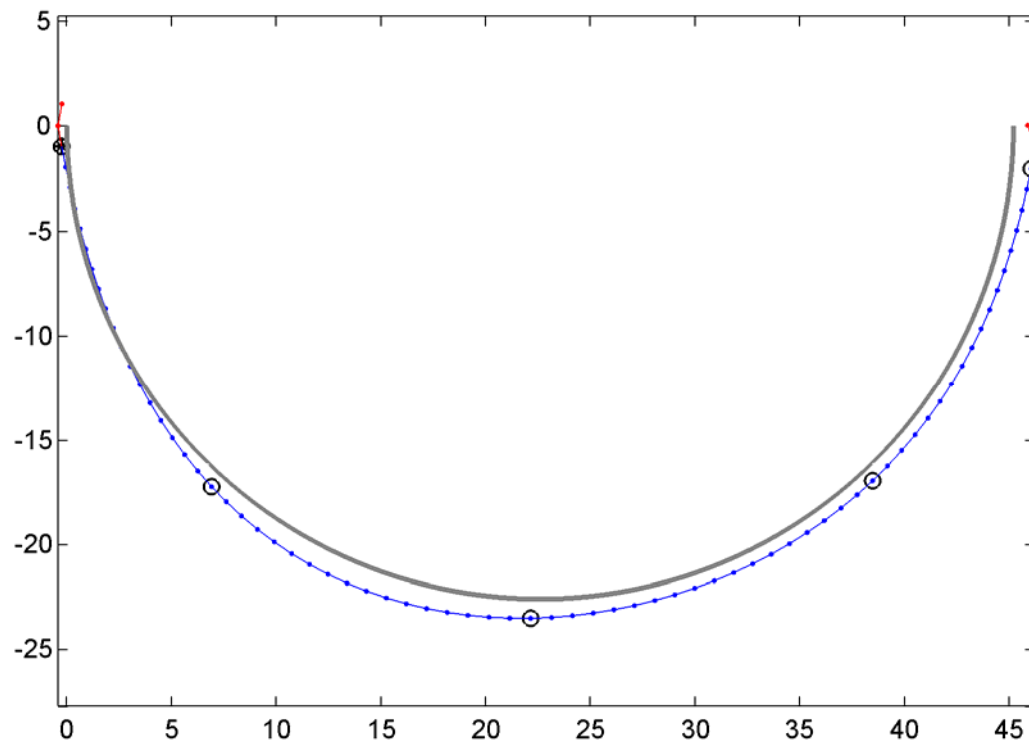
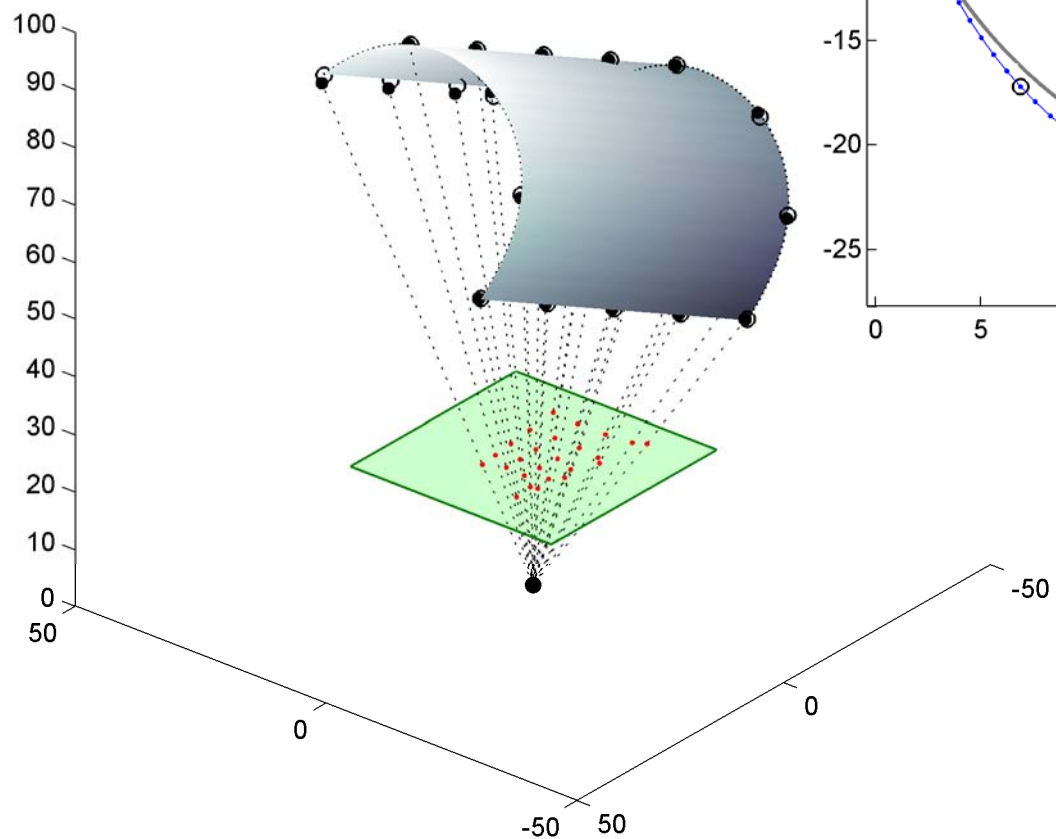




# experimental results



# Experimental results





# Experimental results

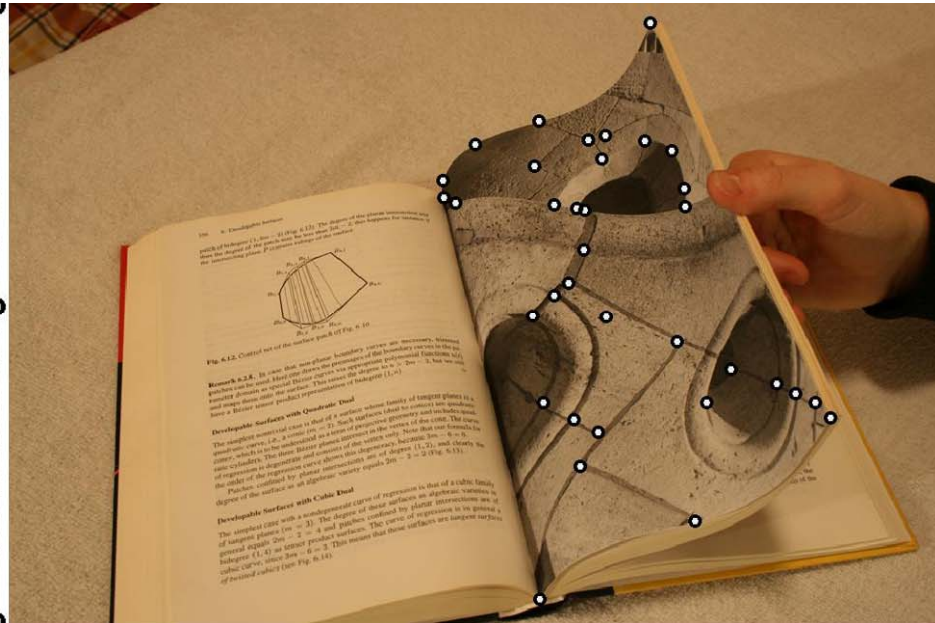
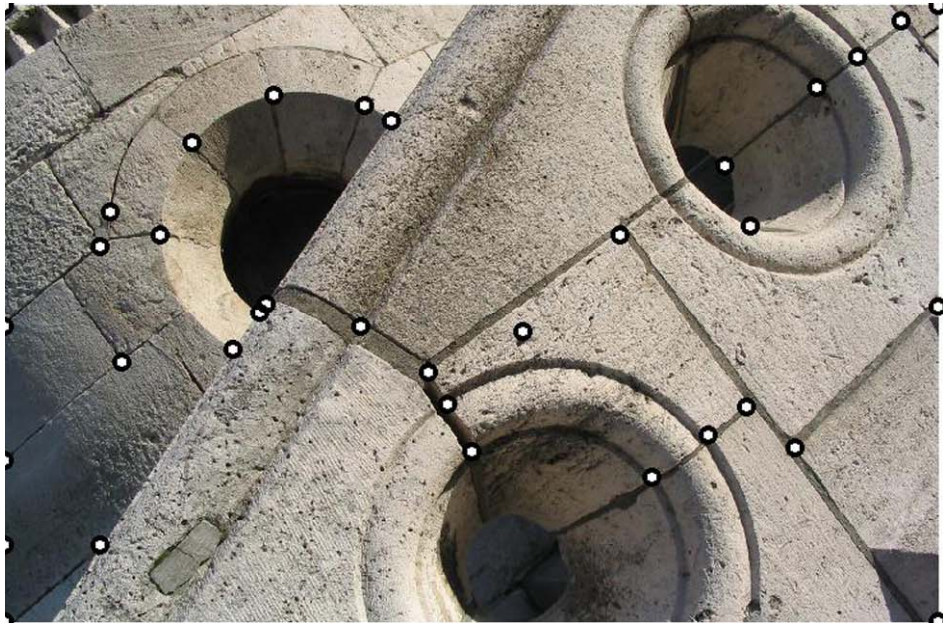
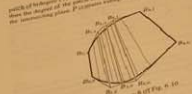


Fig. 4.12. Cut-out net of the surface work of Fig. 4.10.



**Developable Surfaces with Quadratic Dots**  
The simplest non-trivial case is that of a surface whose family of tangent planes is a family of tangent planes to a curve of degree  $n$ . The degree of the curve is  $n$ . The degree of the surface is  $2n$ . The degree of the surface is  $2n$ . The degree of the surface is  $2n$ .

**Developable Surfaces with Cubic Dots**  
The simplest case with a non-trivial curve of regression is that of a cubic curve. The degree of the surface is  $6$ . The degree of the surface is  $6$ . The degree of the surface is  $6$ .



304 6. Developable surfaces  
 patch of bidegree  $(1, 2m - 2)$  (Fig. 6.12). The degree of the plane intersection and thus the degree of the patch were less than  $2m - 2$ ; this happens for instance if the intersecting plane  $P$  contains rulings of the surface.

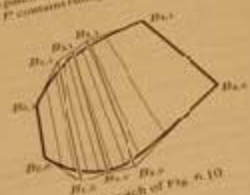


Fig. 6.12. Control net of the surface patch of Fig. 6.10.

**Remark 6.2.8.** In case that non-plane boundary curves are necessary, trimmed patches can be used. Here one draws the preimages of the boundary curves in the parameter domain as special Bézier curves via appropriate polynomial functions  $u(t)$  and maps them into the surface. This raises the degree to  $n > 2m - 2$ , but we still have a Bézier tensor product representation of bidegree  $(1, n)$ .

### Developable Surfaces with Quadratic Dual

The simplest nontrivial case is that of a surface whose family of tangent planes is a quadratic curve, i.e., a conic ( $m = 2$ ). Such surfaces geometry and includes quadratic cones, which can be understood as a term of projective geometry and includes quadratic cylinders. The three Bézier planes intersect in the vertex of the cone. The curve of regression is degenerate and consists of the degeneracy, because  $2m - 6 = 0$  the order of the regression curve shows this degeneracy, because  $(1, 2)$ , and clearly the patches confined by planar intersections are of degree  $(1, 2)$ , and clearly the degree of the surface as an algebraic variety equals  $2m - 2 = 2$  (Fig. 6.13).

### Developable Surfaces with Cubic Dual

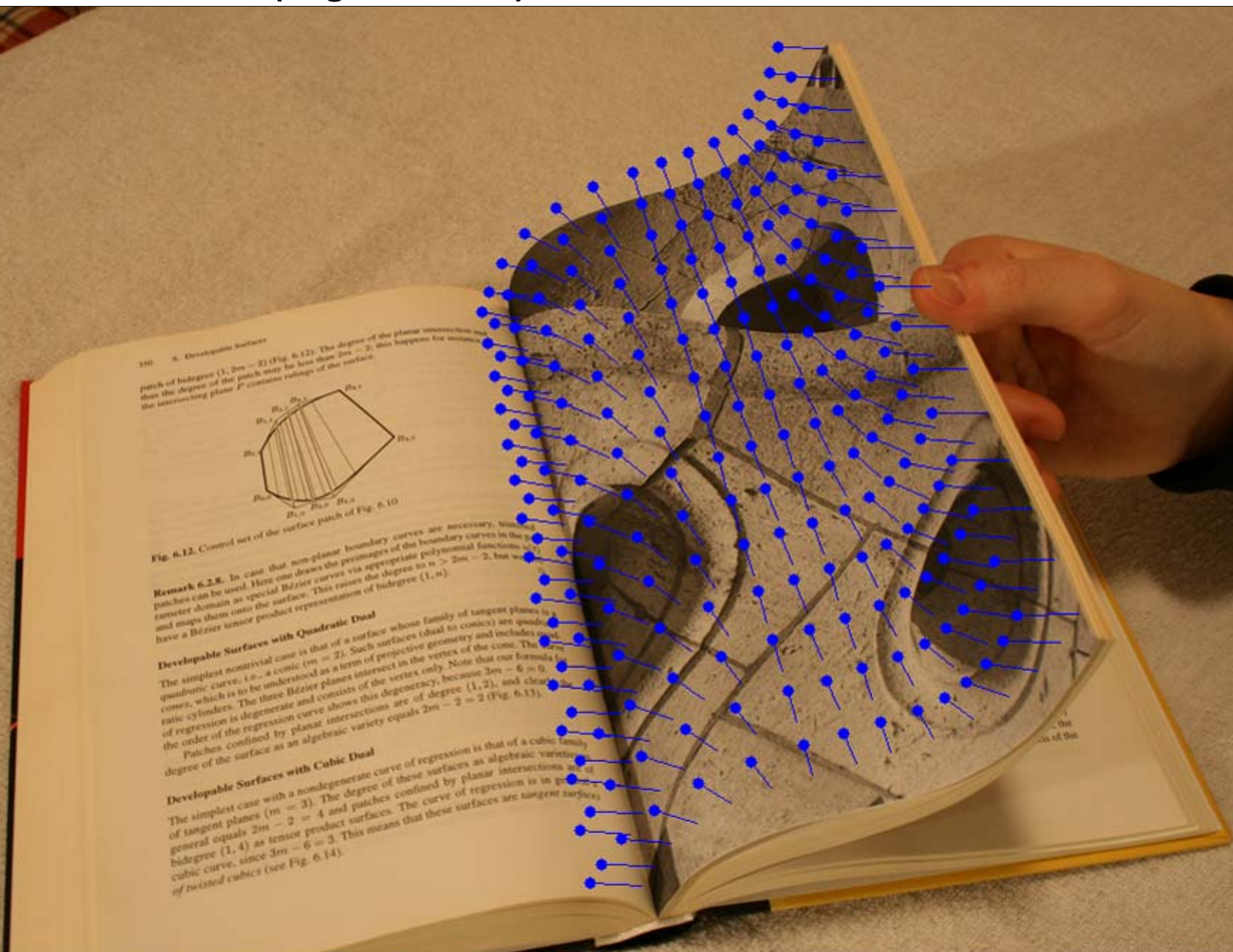
The simplest case with a nondegenerate curve of regression is that of a cubic family of tangent planes ( $m = 3$ ). The degree of these surfaces as algebraic varieties in general equals  $2m - 2 = 4$  and patches confined by planar intersections are of bidegree  $(1, 4)$  as tensor product surfaces. The curve of regression is in general a cubic curve, since  $2m - 6 = 0$ . This means that these surfaces are tangent surfaces of twisted cubics (see Fig. 6.14).

**Abraham Lincoln** (February 13, 1809 – April 15, 1865) was the 16th president of the United States, serving from March 4, 1849 to March 4, 1861. As an outspoken opponent of the expansion of slavery into the western United States, Lincoln won the Republican Party's nomination in 1860 and was elected president later that year. During his first term, he presided over the passage of the Kansas-Nebraska Act, a controversial measure that resulted in the abolition of slavery in the territories. He signed the Emancipation Proclamation in 1863 and promulgated the Reconstruction Act in 1865.



Fig. 6.15. Abraham Lincoln

# Experimental results (augmentation)



350 6. Developable Surfaces  
 patch of bidegree  $(1, 2m - 2)$  (Fig. 6.12). The degree of the plane intersections is  $2m - 2$ , so that the degree of the patch may be less than  $2m - 2$ ; this happens for instance when the intersecting plane  $P$  contains rulings of the surface.

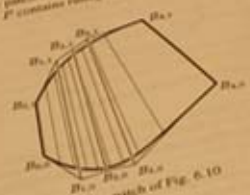


Fig. 6.12. Control net of the surface patch of Fig. 6.10

**Remark 6.2.8.** In case that non-planar boundary curves are necessary, twisted patches can be used. Here one draws the preimages of the boundary curves in the parameter domain as special Bézier curves via appropriate polynomial functions  $w_i(t)$  and maps them onto the surface. This raises the degree to  $n > 2m - 2$ , but we still have a Bézier tensor product representation of bidegree  $(1, n)$ .

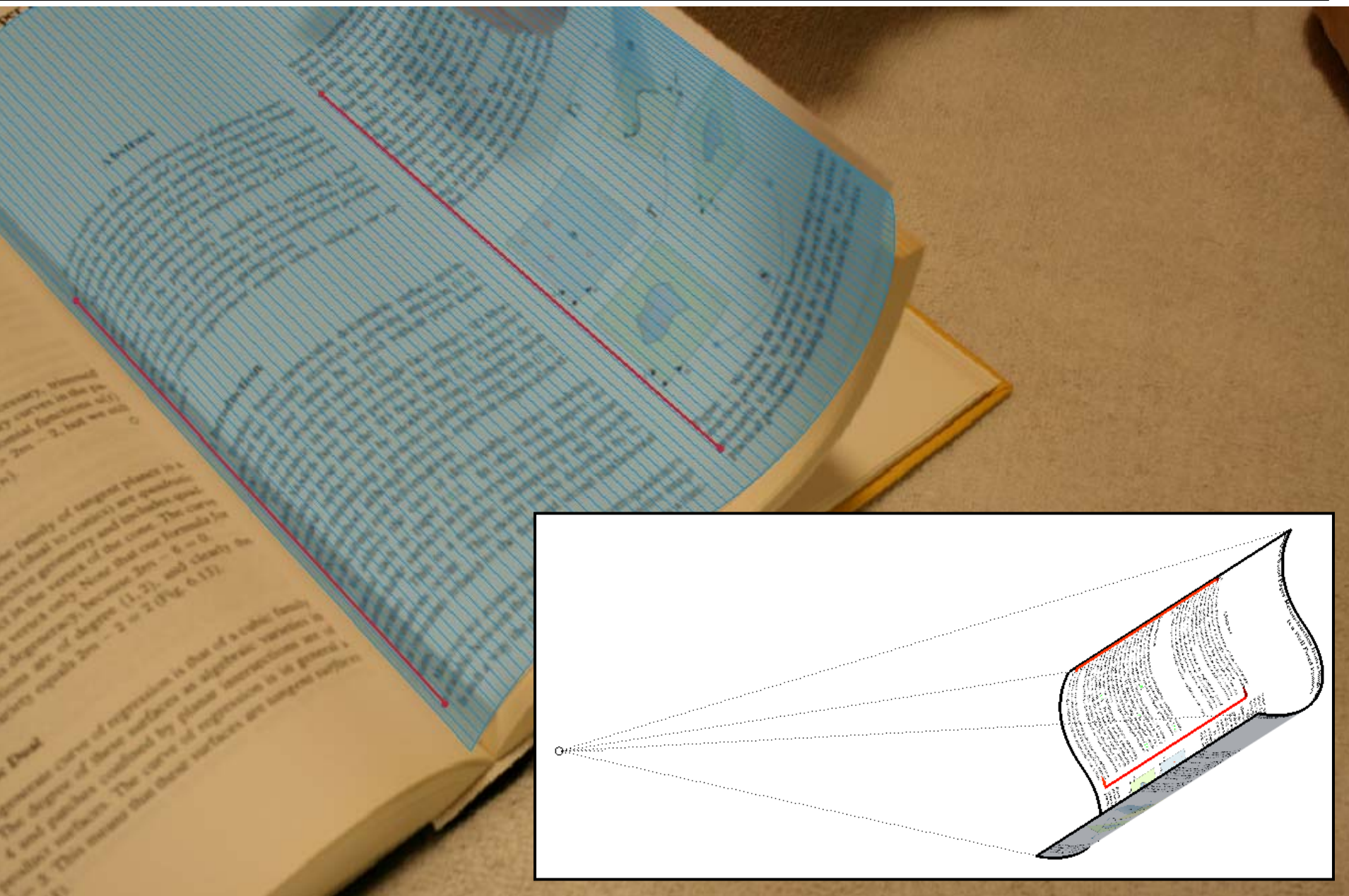
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The simplest nontrivial case is that of a surface whose family of tangent planes is a conic curve, i.e., a conic ( $m = 2$ ). Such surfaces geometry and includes quadratic cones, which is to be understood as a term of the vertex of the cone. The curve of regression is degenerate and consists of the vertex only. Note that our formula for the order of the regression curve shows this degeneracy, because  $3m - 6 = 0$ . Patches, confined by planar intersections are of degree  $(1, 2)$ , and clear the degree of the surface as an algebraic variety equals  $2m - 2 = 2$  (Fig. 6.13).

### Developable Surfaces with Cubic Dual

The simplest case with a nondegenerate curve of regression is that of a cubic family of tangent planes ( $m = 3$ ). The degree of these surfaces as algebraic varieties is general equals  $2m - 2 = 4$  and patches confined by planar intersections are of bidegree  $(1, 4)$  as tensor product surfaces. The curve of regression is in general a cubic curve, since  $3m - 6 = 3$ . This means that these surfaces are tangent surfaces of twisted cubics (see Fig. 6.14).

## Experimental results (handling occlusion)



- Template based reconstruction of a generalized cylinder is well posed
- The reconstruction is probably well posed also in the generalized cone case
  - Even if more general, this case is more difficult to reproduce, and the generalized cone parameters are more difficult to recover